Practical Relative Degree Concept

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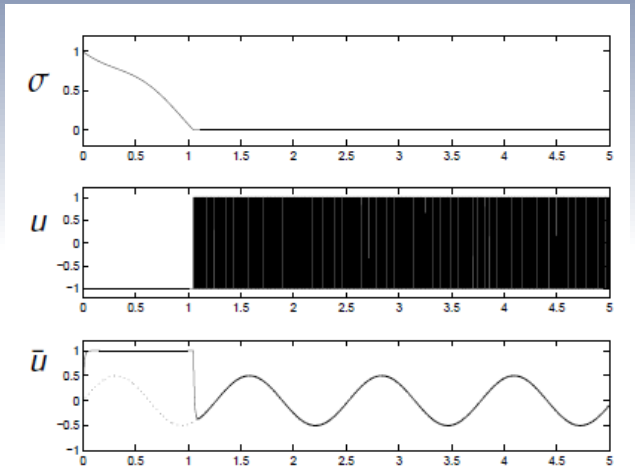
The simplest example

$$\dot{\sigma} = \alpha + u = \alpha - \text{sign}(\sigma), \quad \sigma(0) = 1$$

with $\alpha \in (-1, 1)$.

- $\sigma > 0 \Rightarrow \dot{\sigma} =< 0$
- $\sigma < 0 \Rightarrow \dot{\sigma} => 0$

and $\sigma(t) \equiv 0, \forall t \geq T$.



Remark

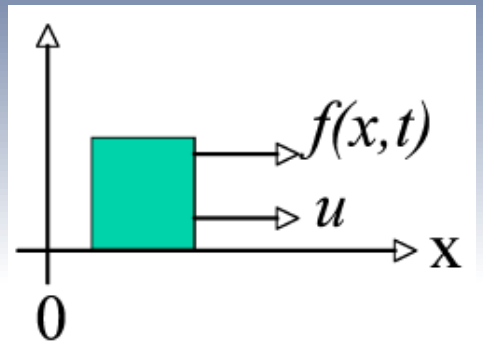
- $0 = \alpha - \text{sign}(0)$?
- The right-hand side is discontinuous.
- After arriving to $\sigma = 0$, sliding along $\sigma \equiv 0$.
- Finite-time convergence.
- Differential inclusion.

$$\dot{\sigma} \in [-\alpha, \alpha] - \text{sign}(\sigma)$$

Mechanical system

A generic mechanic system

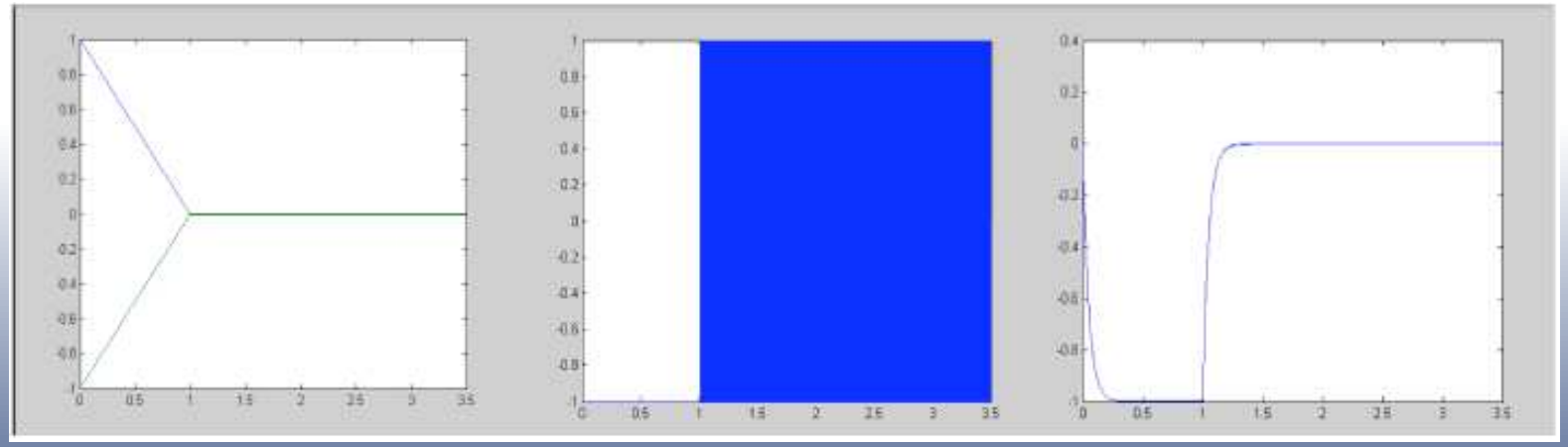
$$\begin{cases} \dot{x}_1 = & x_2 \\ \dot{x}_2 = & u + f(t,x) \\ \sigma = & x_2 \end{cases}$$



with σ as output and select

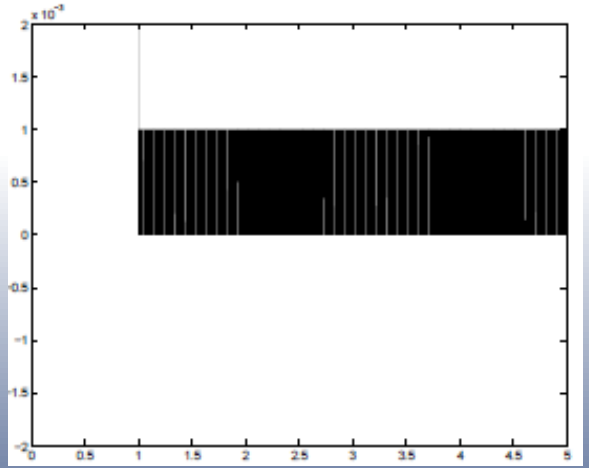
$$u = -\text{sign}(\sigma) = \text{dry friction}$$

x_1 : position.
x_2 : velocity.
σ : measurement



Summary: Late 50 th

- Certainly we stopped, but where?
- No control over x_1 (position)
- Can we manipulate both x_1 and x_2 at the same time?
- Definition of solution on the discontinuity surface is needed
- High frequency discontinuous (switching) control
- Chattering



Chattering as the relative degree problem

Figure: Prof. Levant and Prof. Fridman

Second Order Sliding Modes

$$\begin{cases} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= u + f(x,t) \\ \sigma &= x_1 \end{cases}$$

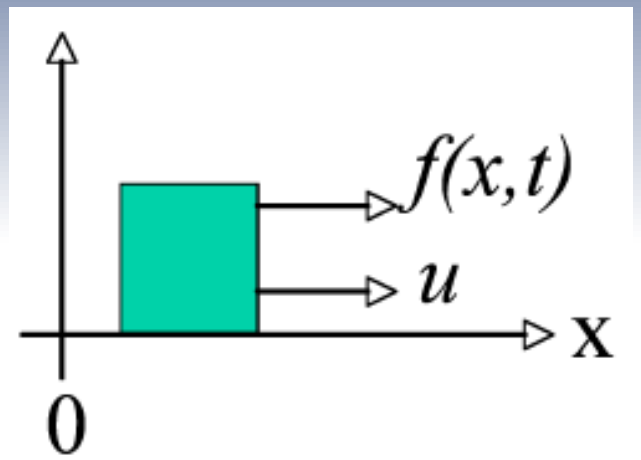


Figure: Prof. Emelyanov,Prof. Korovin and Prof. Levant.

- $f(x,t)$ unknown uncertainties/perturbations.
- All the partial derivatives of $f(x,t)$ are bounded on compacts

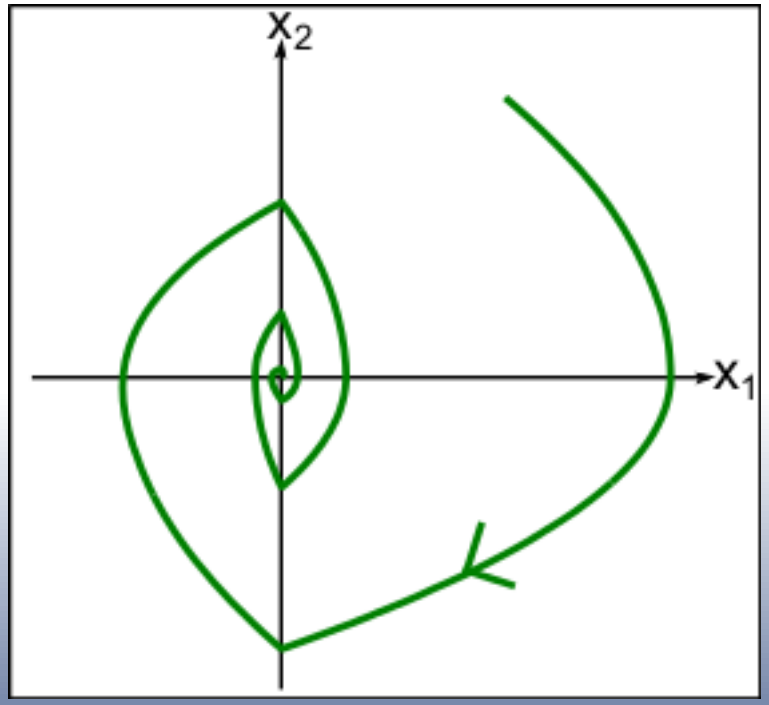
Main Objective

To design a control u such that the origin of system is finite-time stable, in spite of the uncertainties/perturbations $f(x,t)$, with $|f(x,t)| < f^+$ for all t, x

Twisting algorithm

$$u = -a\,\mathrm{sign}(x_2) - b\,\mathrm{sign}(x_1),\ \ b > a + f^+, a > f^+.$$

- Known bounds f^+
- a and b chosen appropriately (Emelyanov et al. 86),
- Ensures finite-time exact convergence for both x_1 and x_2



Grey-box Control via Sliding Mode Control

Arbitrary Order Sliding Mode Controllers

$$\dot{X} = F(t, X) + G(t, X)u, X \in R^n, u \in R$$
$$\sigma = \sigma(X, t), \in R.$$

- σ has a fixed and known relative degree r.
- Control problem is transformed into the finite-time stabilization of an uncertain differential equation

$$\sigma^{(r)} = f(t, X) + g(t, X)u, \tag{1}$$

and corresponding differential inclusion

$$\sigma^{(r)} \in [-C, C] + [K_m, K_M]u, \tag{2}$$

where C, K_m and K_M are known constants.

Nested arbitrary order sliding-mode controllers

- 2001: Nested arbitrary order SM controller
- Solve the finite-time exact stabilization problem for an output with an arbitrary relative degree.
- Bounded Lebesgue measurable uncertainties.
- "Nested" higher order sliding-mode(HOSM) controllers are constructed using a recursion

Figure: Prof. Levant

Nested Third Order Singular Terminal Algorithm

- Third Order

$$u = -\alpha \operatorname{sign}\left(\ddot{\sigma} + 2(|\dot{\sigma}|^3 + |\sigma|^2)^{\frac{1}{6}} \times \operatorname{sign}(\dot{\sigma} + |\sigma|^{\frac{2}{3}} \operatorname{sign}(\sigma))\right)$$

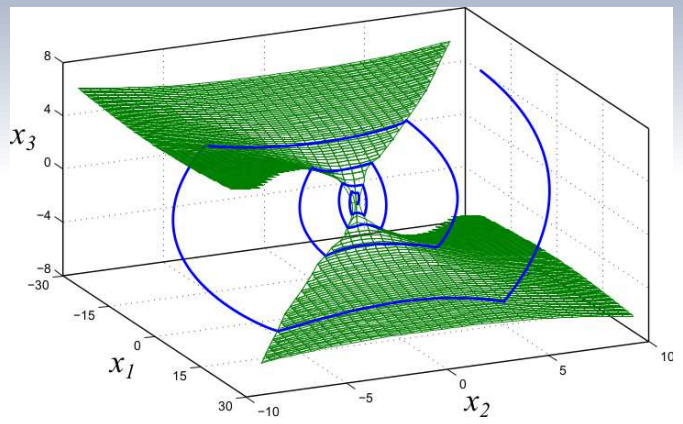


Figure: 3rd Order Nested SM

- Fourth Order

$$u = -\alpha \operatorname{sign}\left(\dddot{\sigma} + 3(\ddot{\sigma}^6 + \dot{\sigma}^4 + \sigma^3)^{\frac{1}{12}} \times \operatorname{sign}\left(\ddot{\sigma} + (\dot{\sigma}^4 + |\sigma|^3)^{\frac{1}{6}} \operatorname{sign}(\dot{\sigma} + 0.5|\sigma|^{\frac{3}{4}} \operatorname{sign}(\sigma))\right)\right)$$

- Finite-time stabilization of $\sigma = 0$ and its successive derivatives up to $r - 1$.

HOSM Differentiator

- The Nested Controller needs the output and its successive derivatives
- Instrument: HOSM arbitrary order differentiator
- Let $\sigma(t)$ signal to be differentiated $k-1$ times
- Assume that $|\sigma^{(k)}| \leq L$.
- 2-th order HOSM differentiator

$$\begin{aligned}
\dot{z}_0 &= v_0 = -3L^{\frac{1}{4}}\,|z_0-\sigma|^{\frac{3}{4}}\,\operatorname{sign}(z_0-\sigma)+z_1,\\
\dot{z}_1 &= v_1 = -2L^{\frac{1}{3}}\,|z_1-v_0|^{\frac{2}{3}}\,\operatorname{sign}(z_1-v_0)+z_2,\\
\dot{z}_2 &= v_2 = -1.5L^{\frac{1}{2}}\,|z_2-v_1|^{\frac{1}{2}}\,\operatorname{sign}(z_2-v_1)+z_3\\
\dot{z}_3 &= -1.1L\operatorname{sign}(z_3-v_2)
\end{aligned} \tag{3}$$

- z_i true derivative $\sigma^{(i)}(t)$.

Discussion about HOSM

Advantages

- Ensures $\sigma = \dot{\sigma} = \ddot{\sigma} = \cdots = \sigma^{(r-1)} = 0$ in finite time.
- Exact disturbance compensation for SISO systems with relative degree r.
- **Ensures the r-th order precision for the sliding output with respect to the discretization step and fast parasitic dynamics**.
- The sliding surface design is no longer needed.

Disadvantages

- Still produces a discontinuous control signal.
- Anti-chattering strategy: Reconstruction of the perturbation is needed ($\sigma^{(r)}$)

Implementation of HOSM

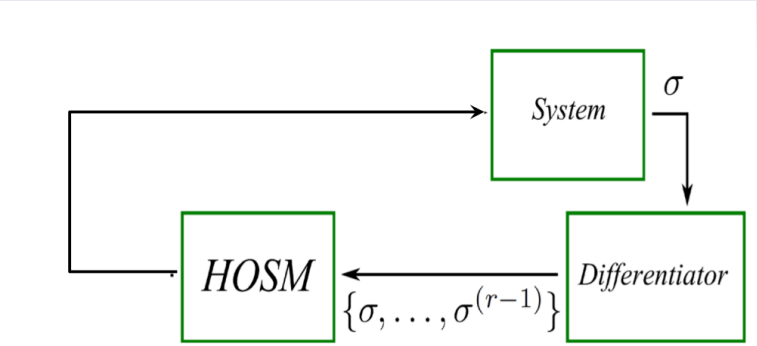


Figure: The implementation of the output-feedback HOSM controller

Practical Relative Degree: Time Domain Approach

Glucose Insulin Regulatory System

- Non linear system.
- Time variable.
- Large inter-patient variability.
- Parameter identification is expensive and invasive for the patient.

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High-Order Sliding-Mode Control

HOSMC can be design as a black-box control where only the relative degree of the system has to be known.

High-Order Sliding Mode Control is insensitive with respect to parameter uncertainties.

Relative Degree

The relative degree r is defined as the order of the total time derivative of σ where the input variable u explicitly appears for the first time.

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Bergman Minimal Model (BeM)

BeM [a]

[a] Bergman et al., 1979

$$\begin{aligned}\dot{B}_1 &= -p_1[B_1 - G_b] - B_1 B_2 + \dot{D}(t)\\ \dot{B}_2 &= -p_2 B_2 + p_3[B_3 - I_b]\\ \dot{B}_3 &= -n[B_3 - I_b] + \gamma[B_1 - h]^+ u(t)\end{aligned}$$

Table: Variables

Variable	Description
B_1	Blood glucose concentration (Output)
B_2	Effect of insulin on glucose uptake
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Glucose Regulation Models

Model	RD	No. States
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Candas-Radziuk	3	4
Cobelli	3	7
Hovorka	5	8
Dalla Man	5	8
Sorensen	5	18

- Output: blood glucose
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Consider the system

$$\dot{x} = a(t,x) + b(t,x)v, \quad \sigma = \sigma(t,x) \tag{2}$$

where $x \in \mathbf{R}^n$, $v \in \mathbf{R}$ is the input, a, b and $\sigma : \mathbf{R}^{n+1} \to \mathbf{R}$ are unknown smooth functions, n can be also uncertain. The output σ is measured in real time.

Assumption 1

Smooth uncertain functions a, b and σ are defined in some open region $\Omega \subset \mathbf{R}^{n+1}$. It is supposed that provided the input v is a Lebesgue-measurable function of time, $|V| \leq v_M$, all solutions starting from an open region $\Omega_x \subset \mathbf{R}^n$ at $t \in t_a$ can be extended in time up to $t = t_b > t_a$ without leaving the region Ω.

Assumption 2

The relative degree r_p of the system is assumed to be constant and known. It means that for the first time the input variable v appears explicitly in the r_pth total time derivative of σ. It can be checked that

$$\sigma^{(r_p)} = h(t,x) + g(t,x)v \tag{3}$$

where $h(t,x) = \sigma^{(r_p)}|_{v=0}, g(t,x) = \frac{\partial}{\partial v}\sigma^{(r_p)}$ are some unknown smooth functions, which can be expressed in the terms of Lie derivatives. The set Ω_x is supposed to contain r_p-sliding points at the time $t = t_a$.

Assumption 3

It is supposed that

$$0 < K_m \leq \frac{\partial}{\partial v}\sigma^{(r_p)} \leq K_M, |\sigma^{(r_p)}|_{v=0}| \leq C \tag{4}$$

hold in Ω for some K_m, K_M, $C > 0$. Conditions (4) are formulated in terms of input-output relations.

Assumption 3

Let the unaccounted-for dynamics is described by the equations

$$\mu \dot{z} = f(z, u), v = v(z) \tag{5}$$

where $z \in \mathbf{R}^m$, $u \in \mathbf{R}$ is the control and the input of the unaccounted-for dynamics, output $v(z)$ is continuous and $f(z, u)$ is a locally bounded Borel-measurable function, the time constant $\mu > 0$ is a small parameter. All differential equations are understood in the Filippov sense.

Assumption 3

The control u is determined by a feedback

$$u = U(\sigma, \dot{\sigma}, \ldots, \sigma^{(r-1)}) \tag{6}$$

where U is a function continuous almost everywhere, and bounded by some constant u_M, $u_M > 0$, in its absolute value. Being applied directly to $\dot{x}$, i.e. with

$$v = u, \tag{7}$$

it is supposed to locally establish the r_p-sliding mode $\sigma \equiv 0$.

Assumption 4

Initial values of z belong to some compact region Ω_{z0}. The unaccounted-for dynamics is assumed to be Bounded-Input-Bounded-State (BIBS), with $\mu = 1$. Since $|u| \leq u_M$, this provides for the infinite extension in time of any solution of u and for z belonging to another compact region Ω_z independent of μ. Indeed, μ can be excluded by the time transformation $\tau = t/\mu$. This assumption causes also the "internal" output v to be bounded in its absolute value by some constant $v_M > u_M > 0$.

Assumption 5

The dynamic output feedback u is supposed to be r_p-sliding homogeneous, which means that the identity

$$U(\sigma, \dot{\sigma}, \ldots, \sigma^{(r_p-1)}) \equiv U(\kappa^{r_p}\sigma, \kappa^{(r_p-1)}\dot{\sigma}, \ldots, \kappa\sigma^{(r_p-1)}) \tag{8}$$

is kept for any $\kappa > 0$. It is also assumed that the control function U is locally Lipschitz everywhere except a finite number of smooth manifolds comprising a closed set Γ in the space with coordinates $\Sigma = (\sigma, \dot{\sigma}, \ldots, \sigma^{(r_p-1)})$. Note that, due to the homogeneity property (8), the set Γ contains the origin $\Sigma = 0$, where the function U is inevitably discontinuous.

Quasi Continuous HOSMC satisfies Assumption 5. As follows from $\sigma^{(r_p)}$,

$$\sigma^{(r_p)} \in [-C, C] + [K_m, K_M]v. \tag{9}$$

Assumption 6

It is assumed that with control u applied directly to inclusion $\sigma^{(r_p)} \in [-C, C] + [K_m, K_M]v$, a finite-time stable inclusion, u, v, $\sigma^{(r_p)}$ is created. Note that the right-hand sides of the differential inclusions are enlarged at the points of the control discontinuity according to the Filippov procedure.

Assumption 7

The unaccounted-for dynamics are assumed exact in the following sense. With $\mu = 1$ and any constant value of u the output v uniformly tends to u. That means that for any $\delta > 0$ there exists $T > 0$ such that with any u, $u = const$, $|u| \leq u_M$, $z(0) \in \Omega_z$, the inequality $|v - u| \leq \delta$ is kept after the transient time T. It is required also that the function $f(z, u)$ in v be uniformly continuous in u, which means that $||f(z, u) - f(z, u + \triangle u)||$ tends to 0 with $\triangle u \to 0$ uniformly in $z \in \Omega_z$, $|u| \leq u_M$.

Assumption 8

It is supposed that the change of u in $v = u$ at the set Γ to

$$v \in \begin{cases} U(\Sigma), & \Sigma \notin \Gamma \\ [-V_M, V_M], & \Sigma \in \Gamma \end{cases} \tag{10}$$

does not destroy the finite-time convergence.

Theorem

Let assumptions 1-8 hold. Then there exist a vicinity Q *of the* r_p-*sliding set in* Ω_x *at* $t = t_a$*, a time moment* $t_1 \in (t_a, t_b)$*, and* $a_0, a_1, \ldots, a_{r-1} > 0$*, such that with sufficiently small* $\mu > 0$ *for any trajectory of* u*,* v*,* u *starting within* Q *at* $t = t_a$ *the inequalities* $|\sigma| < a_0\mu_p^r, |\dot{\sigma}| < a_1\mu^{r_p-1}, \ldots, |\sigma^{(r_p-1)}| < a_{r_p-1}\mu$ *are kept within* $t \geq t_1$*.*

Practical Relative Degree r_p

Definition

The number of derivatives for which the influence of the input is evident.

Practical Relative Degree Identification

STEP TEST

$$\begin{aligned} \sigma^{(r_p)} &= h(t,x) + g(t,x)u_H \\ u_H &= H(x - t_H) \end{aligned}$$

Practical Relative Degree

Identification Method

$$\sigma^{(r_p)} = h(t,x) + g(t,x)H(x - t_H)$$

Conditions

The practical relative degree is identified when the r_p-th derivative of the output:

- if a discontinuity is observed in $\sigma^{(r_p)}$ at $t = t_H$.
- if a change of slope is seen in $\sigma^{(r_p-1)}$ at $t = t_H$ In the scale of the system.

BeM Practical Relative Degree

Figure: BeM output (glucose) and its derivatives

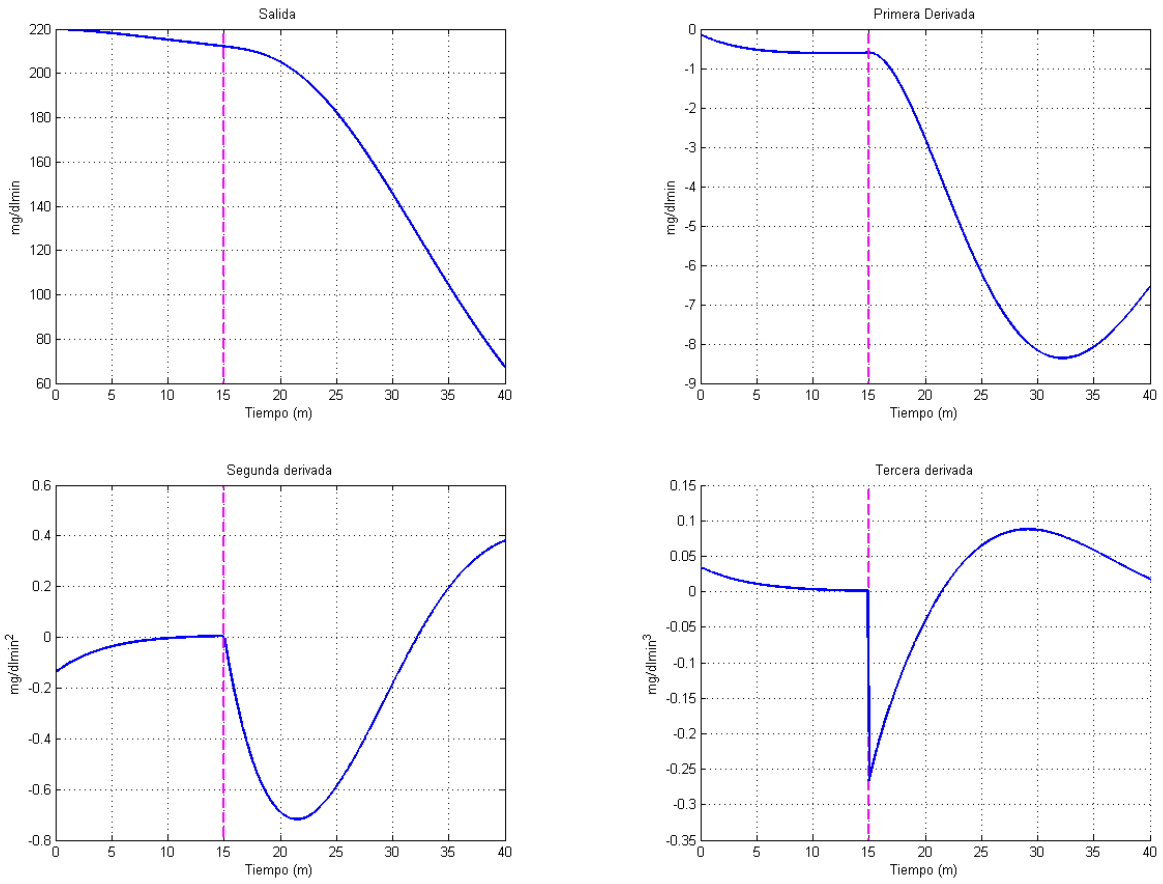
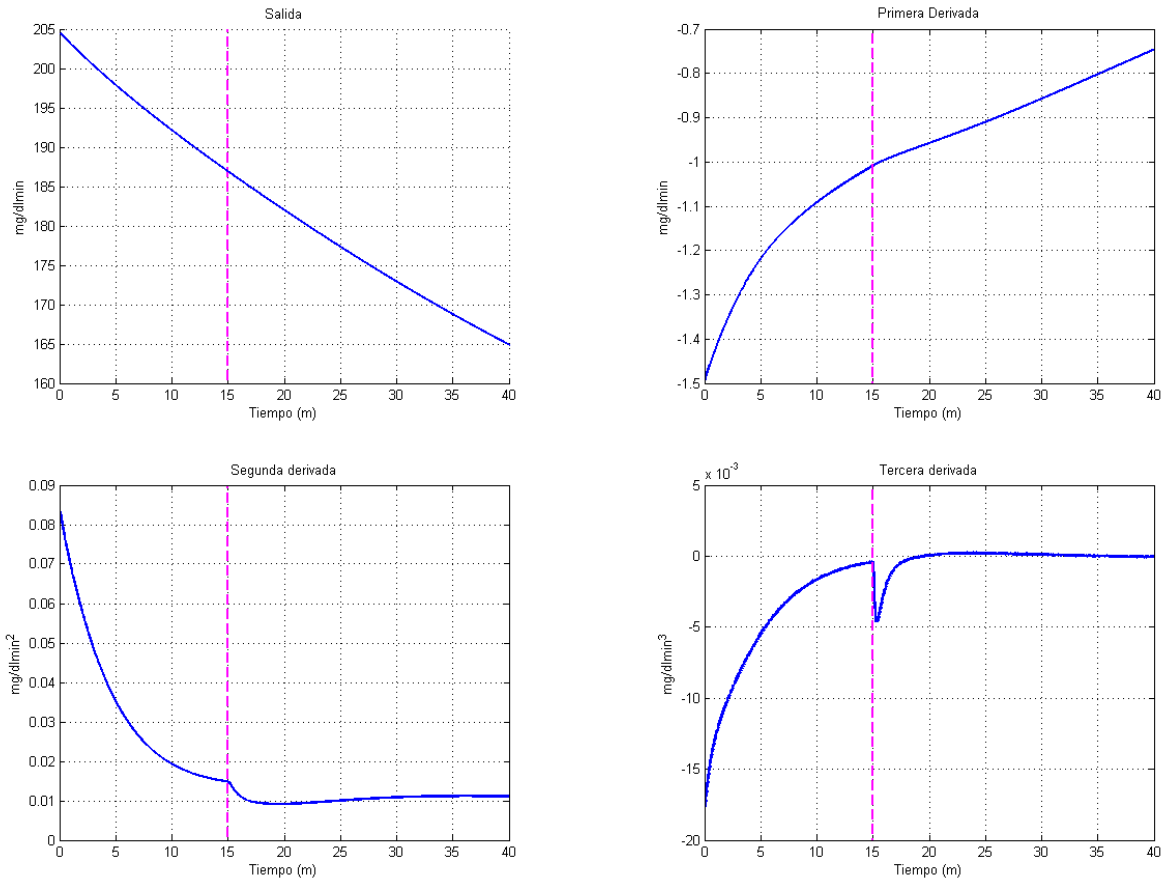


Figure: SoM output (glucose) and its derivatives



Quasi-Continuous HOSM (QC-HOSMC)

Third Order QC-HOSMC

$$u = -\alpha \frac{[\ddot{\sigma} + \beta_2(|\dot{\sigma}| + \beta_1|\sigma|^{2/3})^{1/2}(\dot{\sigma} + \beta_1|\sigma|^{2/3} sign\sigma)]}{[|\ddot{\sigma}| + \beta_2(|\dot{\sigma}| + \beta_1|\sigma|^{2/3})^{1/2}]}$$

QC-HOSMC Simulation for BeM

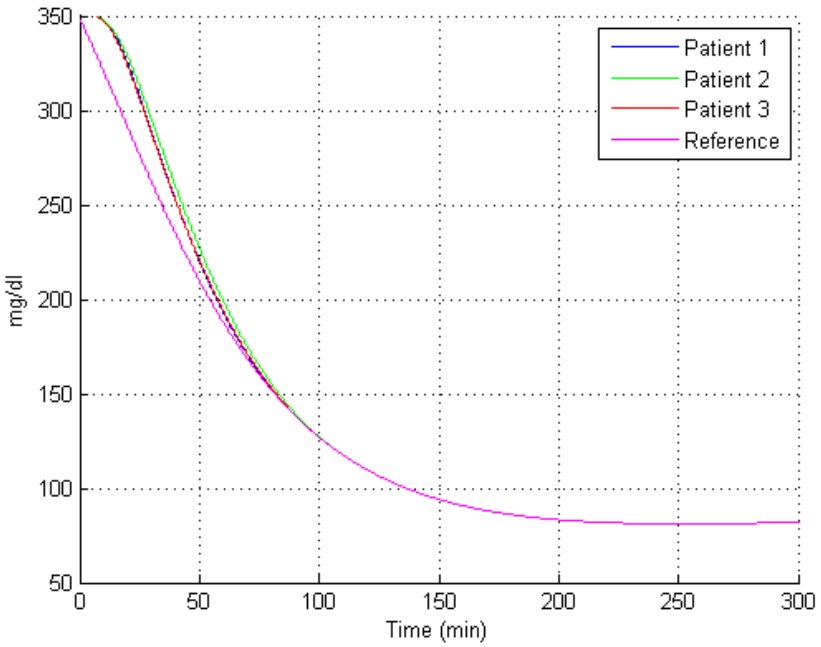


Figure: Glucose concentration for BeM. The simulation represents a postprandial event controlled by a third order QC-HOSMC. No hypoglycemia is observed, and normoglycemia is reached in an acceptable time (100 min). The gains of the controller are the same for the 3 patients.

QC-HOSMC Simulation for SoM[2]

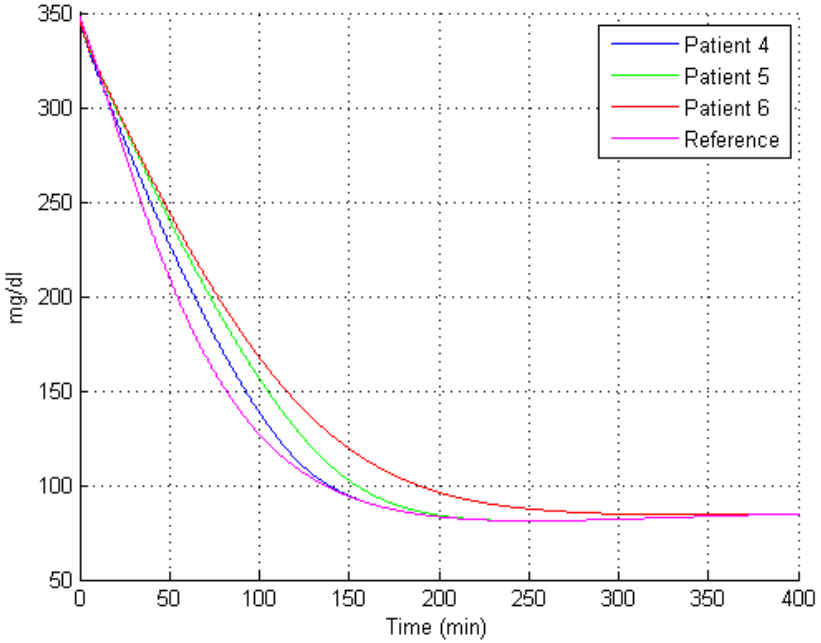
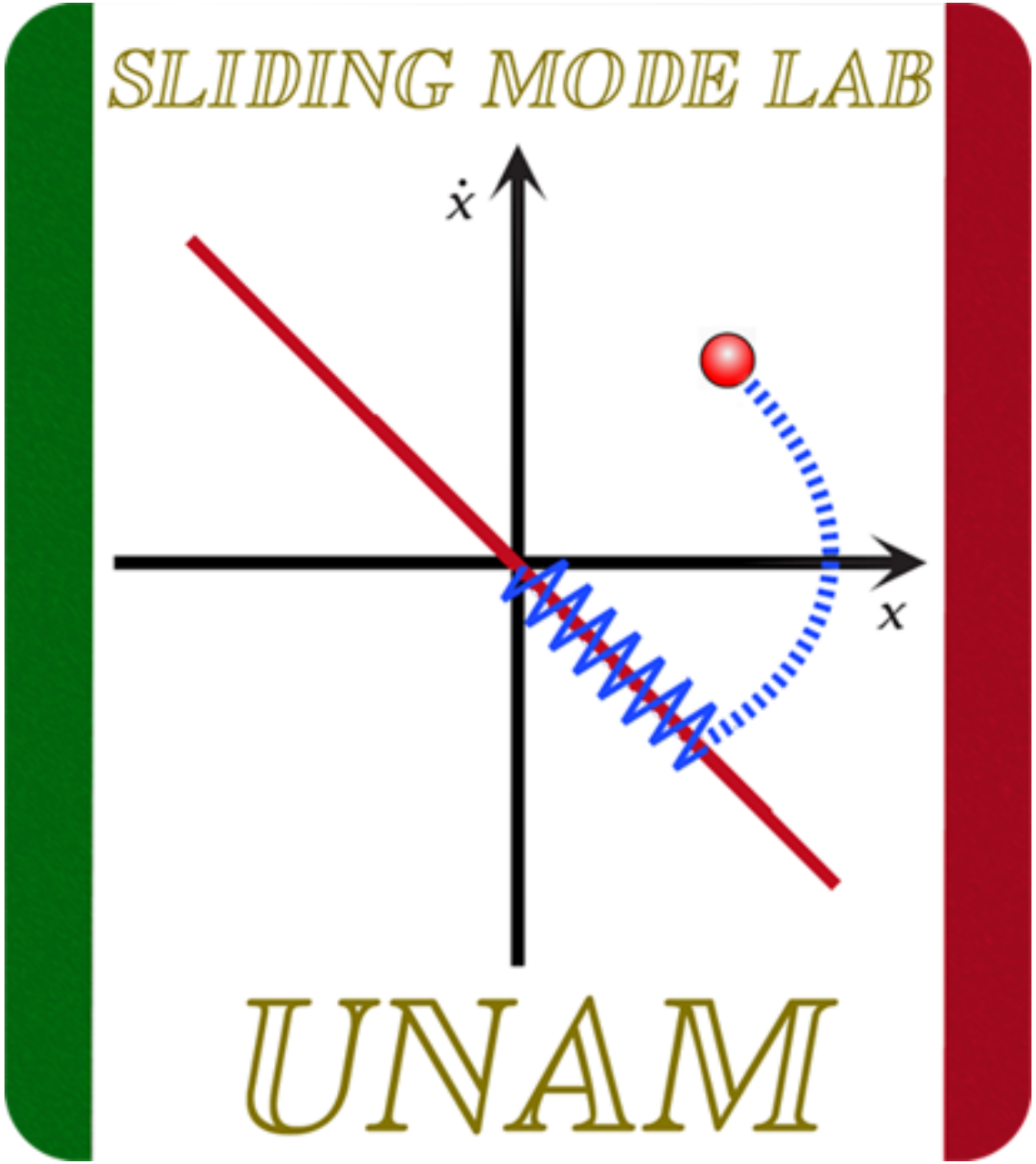


Figure: Glucose concentration for SoM. The simulation represents a postprandial event controlled by a third order QC-HOSMC. No hypoglycemia is observed, and normoglycemia is reached in an acceptable time. The controller is the same as for BeM.

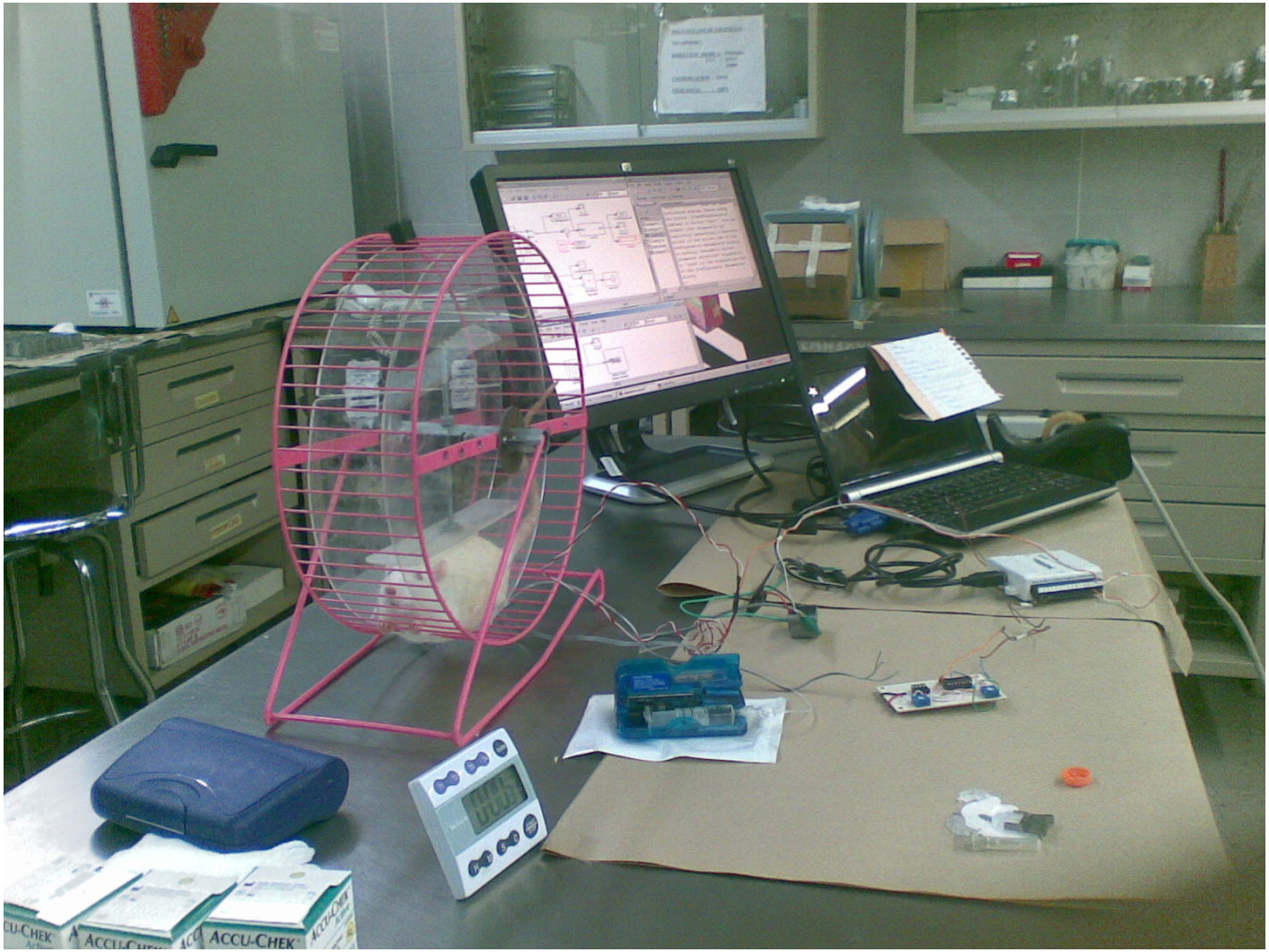
[2] Levant TAC 2005

Conclusions

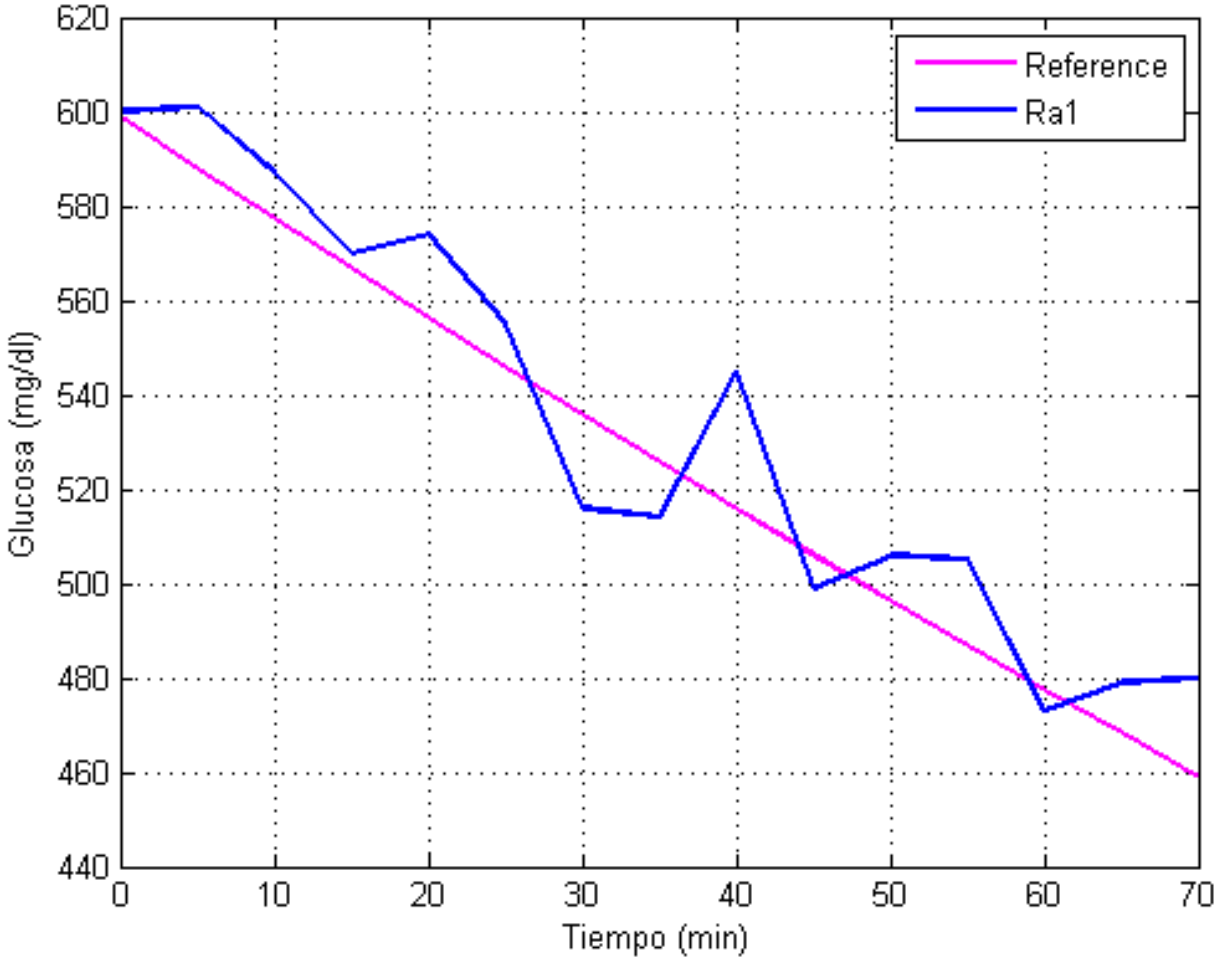
- The **practical relative degree** concept is useful to determinate the order of the HOSMC, for systems where it is not easy to determinate from the analysis of the models.
- HOSMC is suitable to control glucose-insulin regulation system due to it is robust with respect to parametric and relative degree uncertainties.


SLIDING MODE LAB
$\dot{x}$
x
UNAM

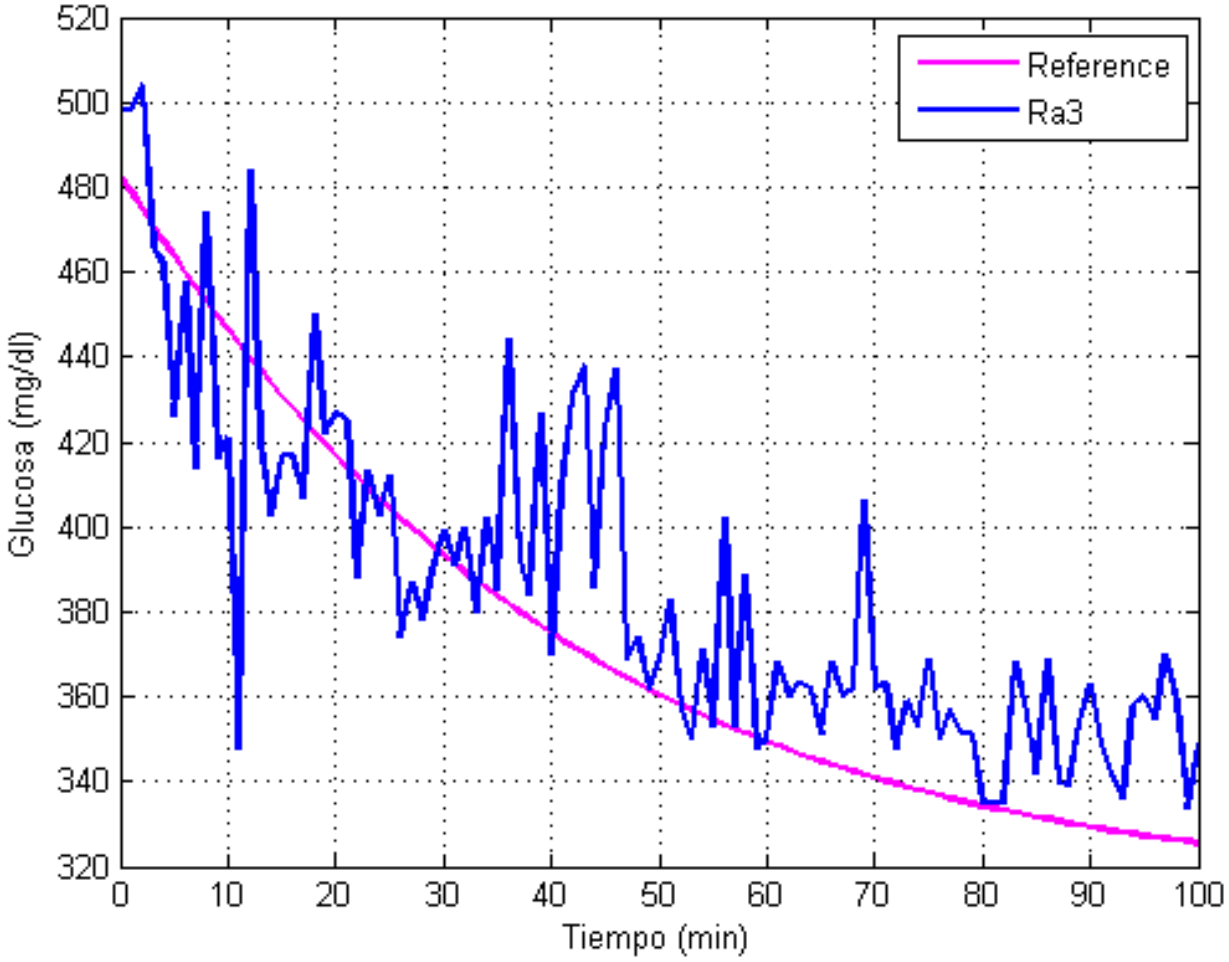
Experimental Set



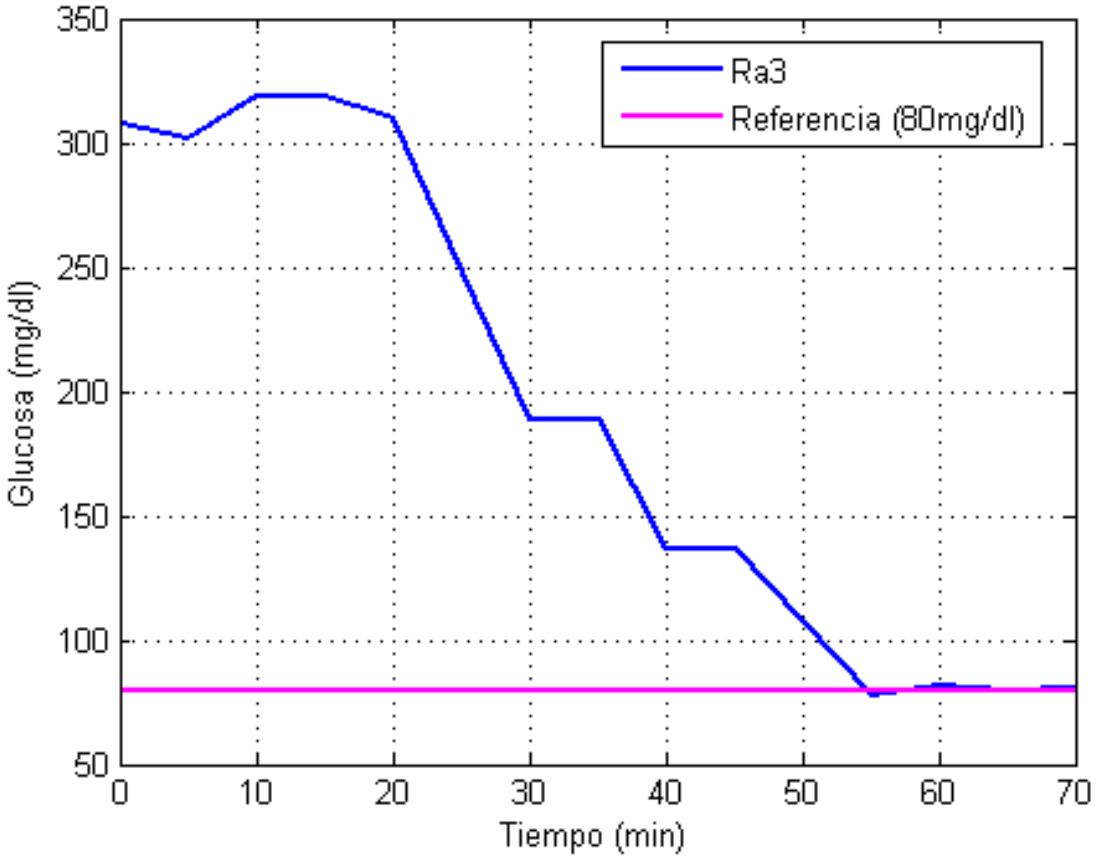
Experiment Ra1



Experiment Ra2

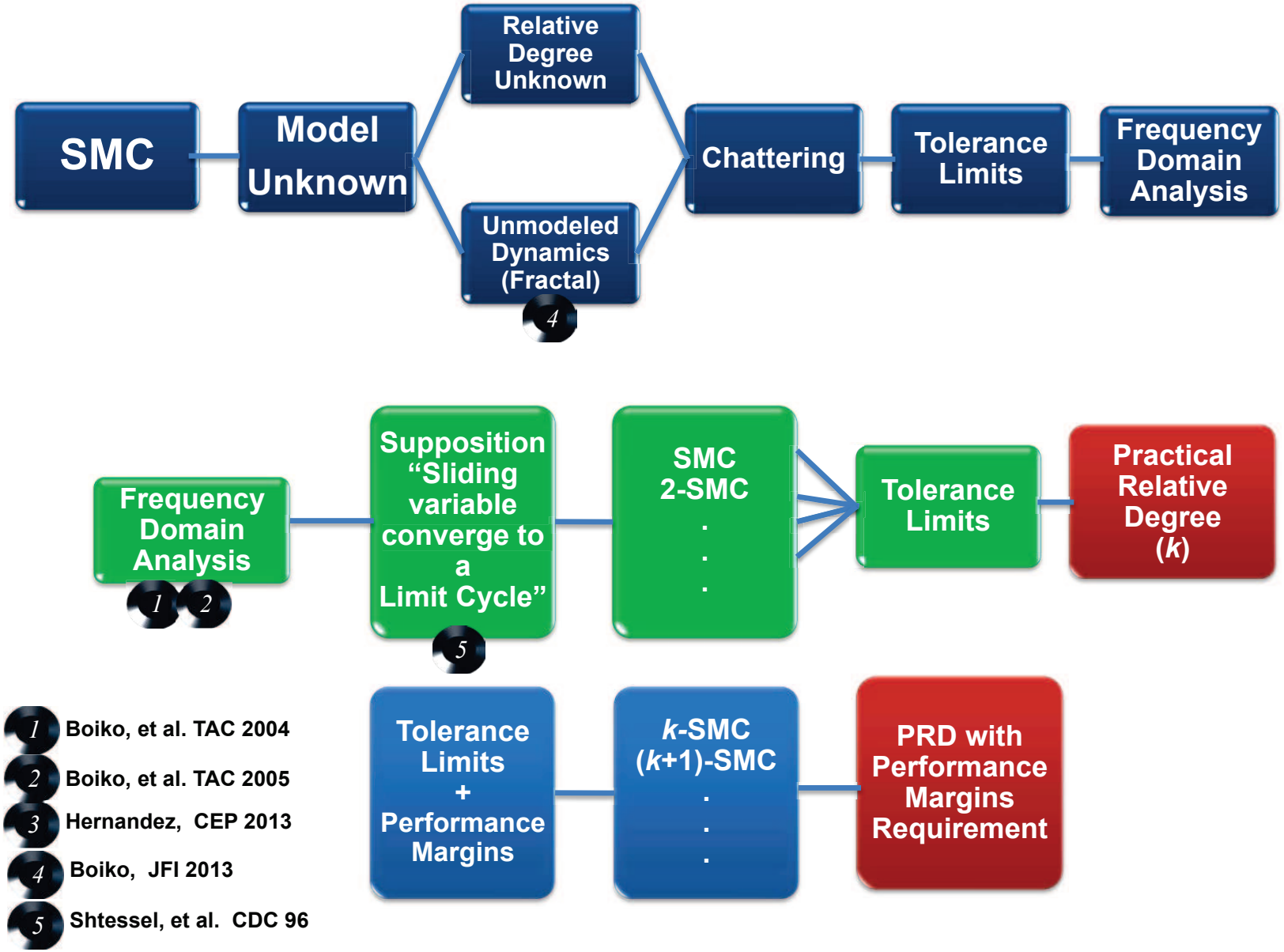


Experiment Ra3



Practical Relative Degree: Frequency Domain Approach

Introduction



1 Boiko, et al. TAC 2004
2 Boiko, et al. TAC 2005
3 Hernandez, CEP 2013
4 Boiko, JFI 2013
5 Shtessel, et al. CDC 96

Problem Statement

Given

- A SISO system

$$\begin{cases} \dot{x} = F(x) + g(x)u \\ \sigma = h(x) \end{cases} \tag{1}$$

  where $x \in R^n$, $\sigma \in R^1$, $F(x), g(x) \in R^n$ are Lipschitz vector fields, and $h(x) \in R$ is a smooth enough function.

- A sliding mode controller (SMC) u

In real systems the unmodeled dynamics have fractal structure [Boiko, JFI2014],

- Relative degree of output is every time infinite.
- Chattering appears [Shtessel, et al. 96],[Boiko, et al. 05].

Problem Statement

Given

- A SISO system

$$\begin{cases} \dot{x} = F(x) + g(x)u \\ \sigma = h(x) \end{cases} \tag{1}$$

  where $x \in R^n$, $\sigma \in R^1$, $F(x), g(x) \in R^n$ are Lipschitz vector fields, and $h(x) \in R$ is a smooth enough function.

- A sliding mode controller (SMC) u

Sliding set will converge to a real sliding mode, where the sliding variable σ **can exhibit self-sustained oscillation** with the finite frequency $0 < \omega < \infty$ and the finite amplitude $A > 0$.

Problem Statement

Given

- A SISO system

$$\begin{cases} \dot{x} = F(x) + g(x)u \\ \sigma = h(x) \end{cases} \tag{1}$$

  where $x \in R^n$, $\sigma \in R^1$, $F(x), g(x) \in R^n$ are Lipschitz vector fields, and $h(x) \in R$ is a smooth enough function.

- A sliding mode controller (SMC) u

The following notions play an important role

- Tolerance Limits
- Practical Relative Degree (PRD), [Levant 2013], [Hernandez, et al. CEP2013]

Problem Statement

Tolerance Limits

The frequency ω_c and amplitude A_c are the tolerance limits of the acceptable limit cycle of the output y, so that self-sustained oscillations of the output y with the amplitudes $A \leq A_c$ and the frequencies $\omega \geq \omega_c$ yield the acceptable performance of the closed loop system in the real sliding mode [Utkin 09]

Problem Statement

Two approaches to PRD notion will be discussed

Based on Tolerant Limits

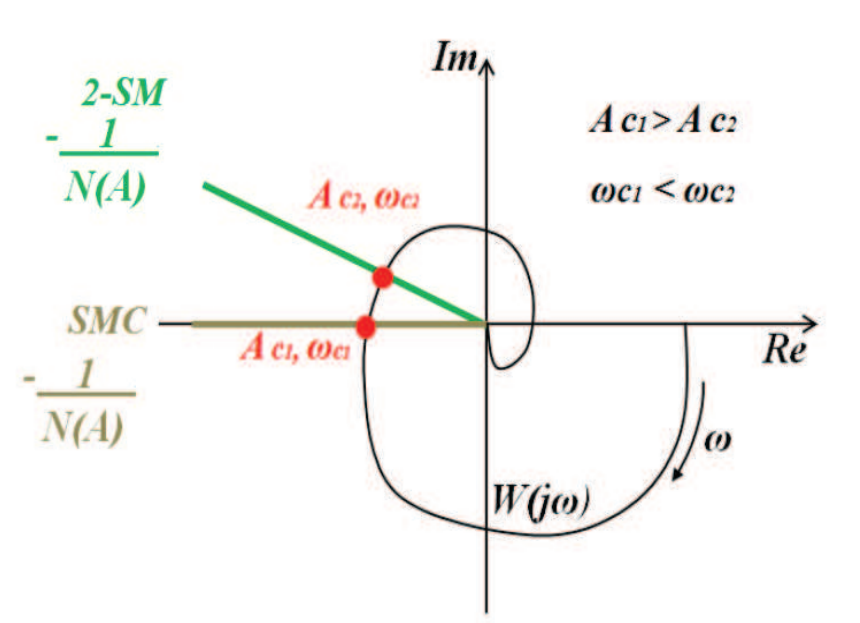
If the model of the system is unknown the PRD of the system is the smallest order of SM controller generating the oscillations in the system satisfying the level of tolerance.

Based on Performance Margins

The performance phase margin (PPM) and performance gain margin (PGM) could be found for each order of sliding mode controllers.
The order of controller satisfying the level of tolerance with desired PGM and PPM could be also considered as PRD.

Motivation Example

Suppose system has $r > 2$ and it is controlled by SMC and 2-SMC



Case 3. Which controller is to be selected for the implementation?

Case\ Controller	SMC	2-SMC
Case 1	$A > A_c$ and/or $\omega < \omega_c$	$A > A_c$ and/or $\omega < \omega_c$
Case 2	$A > A_c$ and/or $\omega < \omega_c$	$A \leq A_c, \omega \geq \omega_c$
Case 3	$A \leq A_c, \omega \geq \omega_c$	$A \leq A_c, \omega \geq \omega_c$

Outline

1. Introduction
   - Problem Statement
2. Practical Relative Degree
3. Performance Margins
   - Definition of Performance Margins
   - Performance margins for conventional SMC
   - Performance margins for 2-SM (Twisting Algorithm)
4. Practical Relative Degree with Performance Margins Requirement
5. Conclusions

Practical Relative Degree

Practical Relative Degree (PRD) is understood as the smallest order r of SMC, that yields a predicted limit cycle in the closed-loop system with the amplitude $A \leq A_c$, $A_c > 0$ and the frequency $\omega \geq \omega_c$, $0 < \omega_c < \infty$.

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Definition of Performance Margins

Performance Phase Margin

The Performance Phase Margin (PPM) in the system is the maximal additional phase shift in $W(j\omega)$ that the closed loop system can tolerate for its output y to exhibit the acceptable limit cycle with $A \leq A_c$, $A_c > 0$ and $\omega \geq \omega_c$, $0 < \omega_c < \infty$ in the real sliding mode.

Performance Gain Margin

The Performance Gain Margin (PGM) in the system is the maximum additional gain in $W(j\omega)$ that the closed loop system can tolerate for its output y to exhibit the acceptable limit cycle with $A \leq A_c$, $A_c > 0$ and/or $\omega \geq \omega_c$ $0 < \omega_c < \infty$ in the real sliding mode.

DF conventional SMC

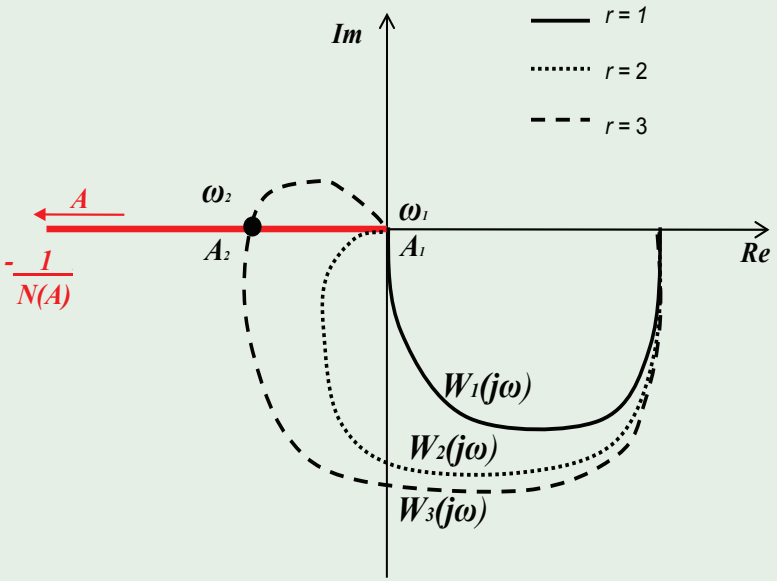
Let the controller be

$$u = -\alpha \text{sign}(y) \tag{2}$$

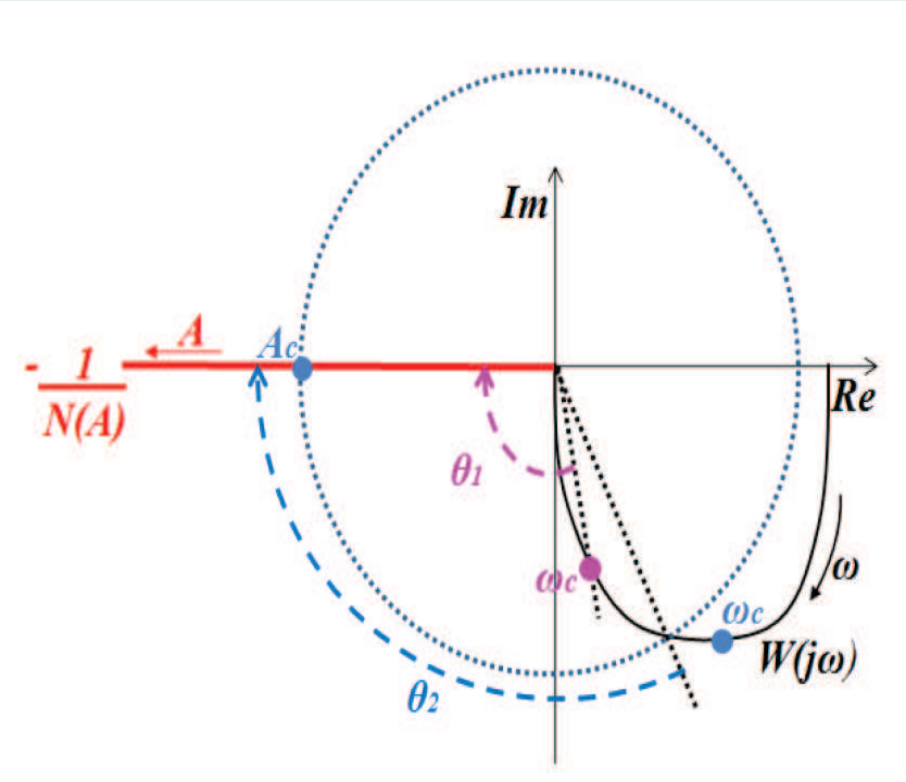
DF is $N(A) = 4\alpha/\pi A$, where A es the amplitude of y.
Harmonic Balance equation is

$$\mathrm{Re}\{W(j\omega)\} + j\,\mathrm{Im}\{W(j\omega)\} = -\frac{\pi A}{4\alpha} \tag{3}$$

where ω is the frequency of the output of system



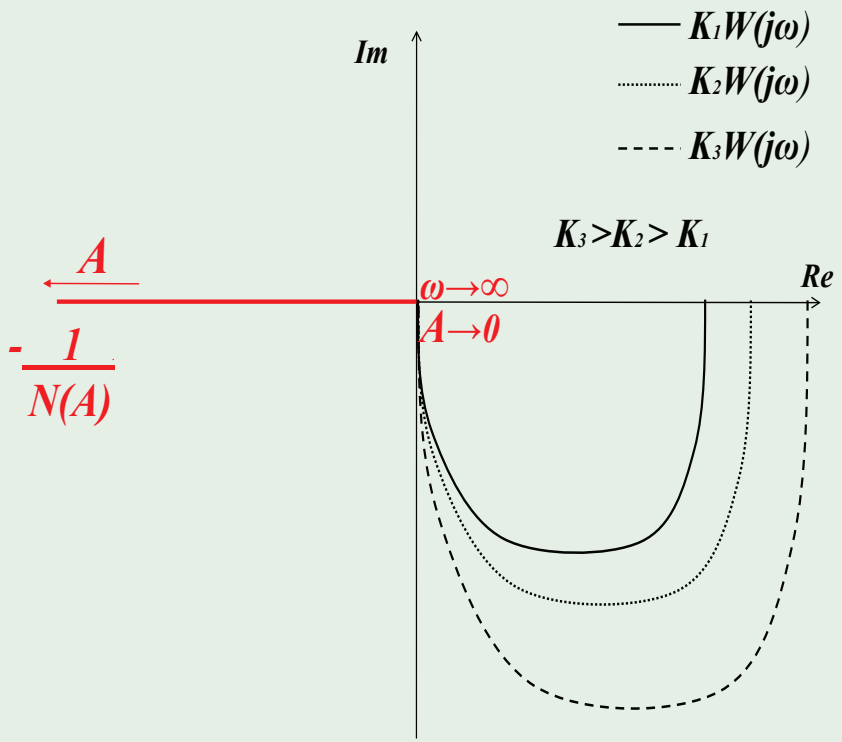
PPM SMC



Performance Phase Margin Method

- 1. Locate maximum amplitude A_c and the minimum frequency ω_c in $-1/N(A)$ and $W(j\omega)$, respectively.
- 3. Plot a circle with radio A_c.
  - 3(a) If the frequency ω_c is located outside of the circle, the PPM should be obtained as the angle formed between the intersection of the circle with $W(j\omega)$ and the negative real axis.
  - 3(b) If the frequency ω_c is located inside of the circle, the PPM should be obtained as the angle formed between the vector associated to ω_c and the negative real axis.

PGM SMC



Performance Gain Margin SMC

- Consider the gain $K > 1$ in the harmonic balance equation

$$K \cdot W(j\omega) = -\frac{1}{N(A)}$$

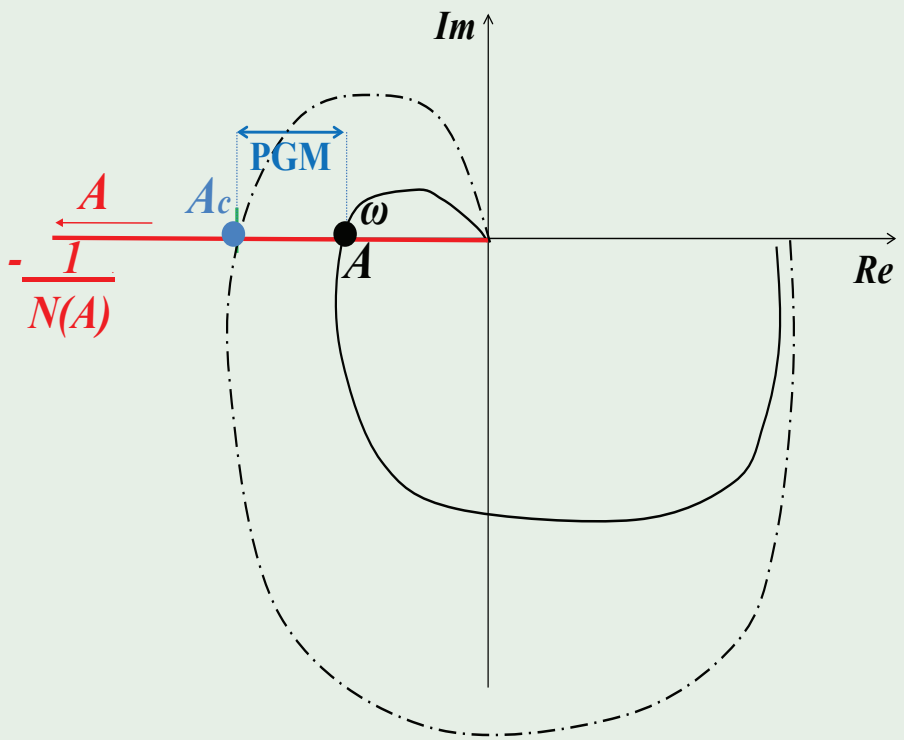
- Solve the harmonic balance equation for K with A_c and ω_c known,

$$\begin{aligned} K\,\mathrm{Re}\,(W(j\omega)) &= -\frac{\pi A}{4\alpha} \\ K\,\mathrm{Im}\,(W(j\omega)) &= 0 \end{aligned}$$

- Value of $K \geq 1$ which satisfy HB equation system with $r \leq 2$ is $K \to \infty$

$$PGM \to \infty.$$

PGM Relay System



PGM Relay System

- 1. Consider the gain $K > 0$ in the harmonic balance equation

$$K \cdot W(j\omega) = -\frac{1}{N(A)}$$

- 2. Solve the harmonic balance equation for K with A_c and ω_c known,

$$K = -\frac{1}{N(A_c)} \frac{1}{W(j\omega_c)}$$

- 3. The value of K is the PGM of the system for the acceptable amplitude A_c.

DF for Twisting Algorithm

Let the controller be

$$u = -c_1 sign(y) - c_2 sign(\dot{y}) \tag{4}$$

where $c_1 > c_2 > 0$.
DF is

$$N(A) = (4/\pi A)(c_1 + jc_2),$$

where A is the amplitude of the output y
Harmonic Balance equation is

$$W(j\omega) = -\frac{A\pi}{4}\frac{c_1 - jc_2}{c_1^2 + c_2^2} \tag{5}$$

where ω is the frequency of the output y.

Performance Margins for 2-SM

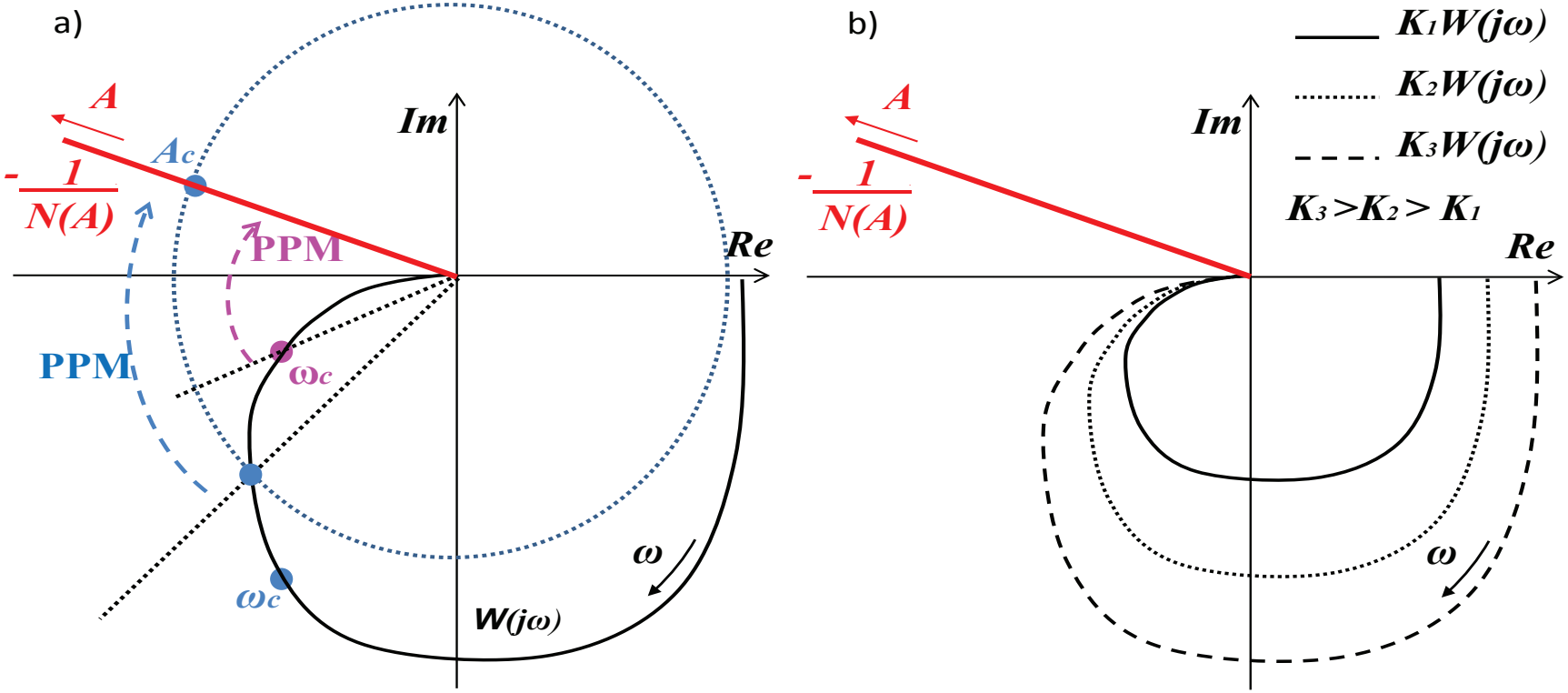


Figura: For Twisting Algorithm. a)PPM, b)PGM

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Practical Relative Degree

Suppose that PPM_c and PGM_c are the acceptable performance phase and gain margins.

Practical Relative Degree

Practical Relative Degree (PRD) is understood as the smallest order r of SMC, that yields a predicted limit cycle in the closed-loop system with the amplitude $A \leq A_c$, $A_c > 0$ and the frequency $\omega \geq \omega_c$, $0 < \omega_c < \infty$ while the $PPM \geq PPM_c$ and $PGM \geq PGM_c$, where PPM_c and PGM_c are the acceptable performance phase and gain margins.

Therefore, the controller, which order corresponds to the calculated PRD, is to be implemented.

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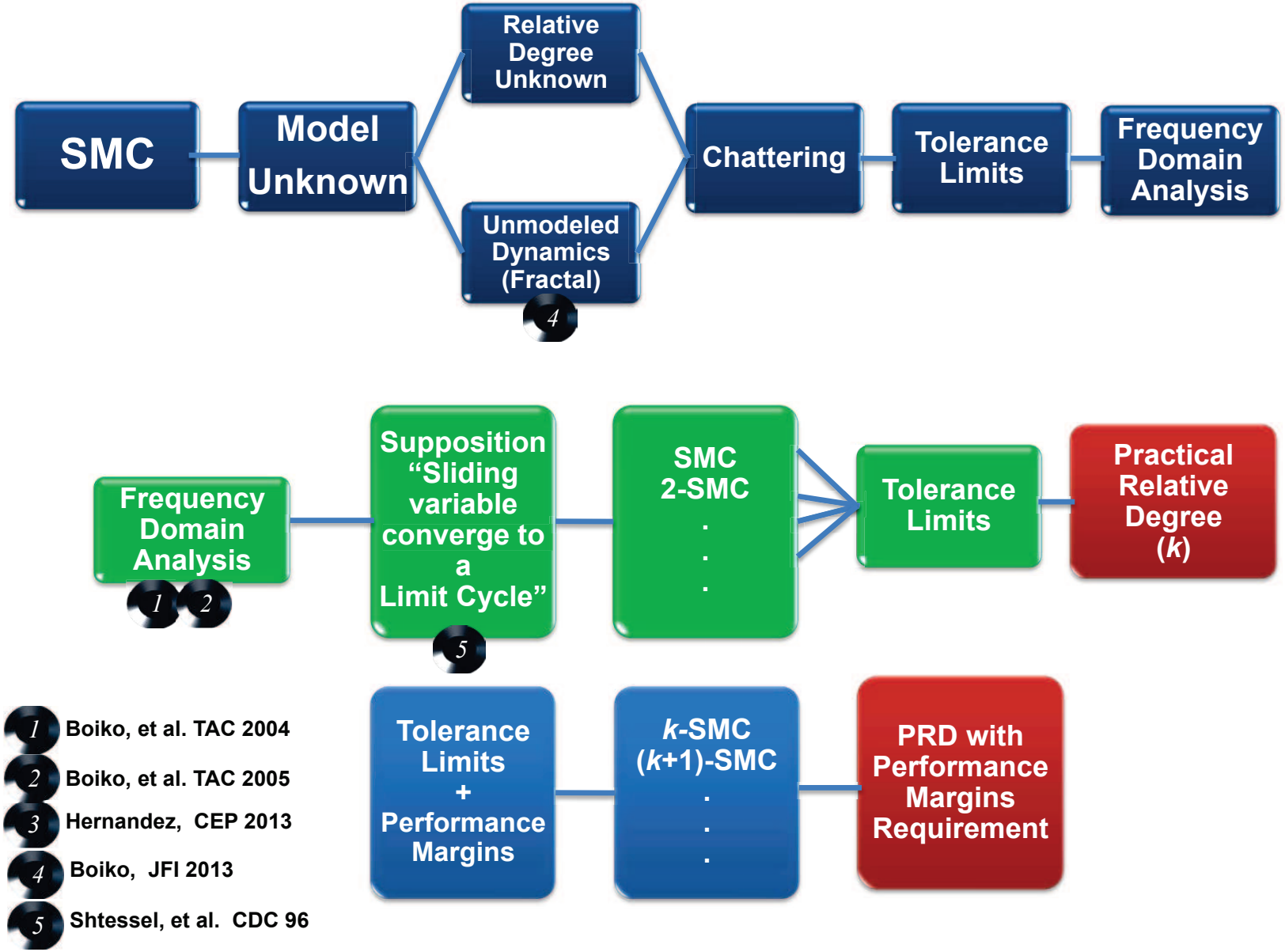
Conclusions

- Practical relative degree in LTI SISO systems controlled by SMC is defined based on tolerance limits in terms of amplitude and frequency of a possible limit cycle on sliding variable.
- PRD is understood as the smallest order of SMC that yields acceptable performance.
- The proposed identification of practical relative degree can be used for the SMC design for systems treated as a black box.
- The notion is given in terms of performance margins which can be useful when the PRD could be affected by parameter changes or errors in the frequency response identification.

Conclusions



1 Boiko, et al. TAC 2004
2 Boiko, et al. TAC 2005
3 Hernandez, CEP 2013
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5 Shtessel, et al. CDC 96