Output robust control of systems subjected to perturbations via high order sliding mode observers

August 2015

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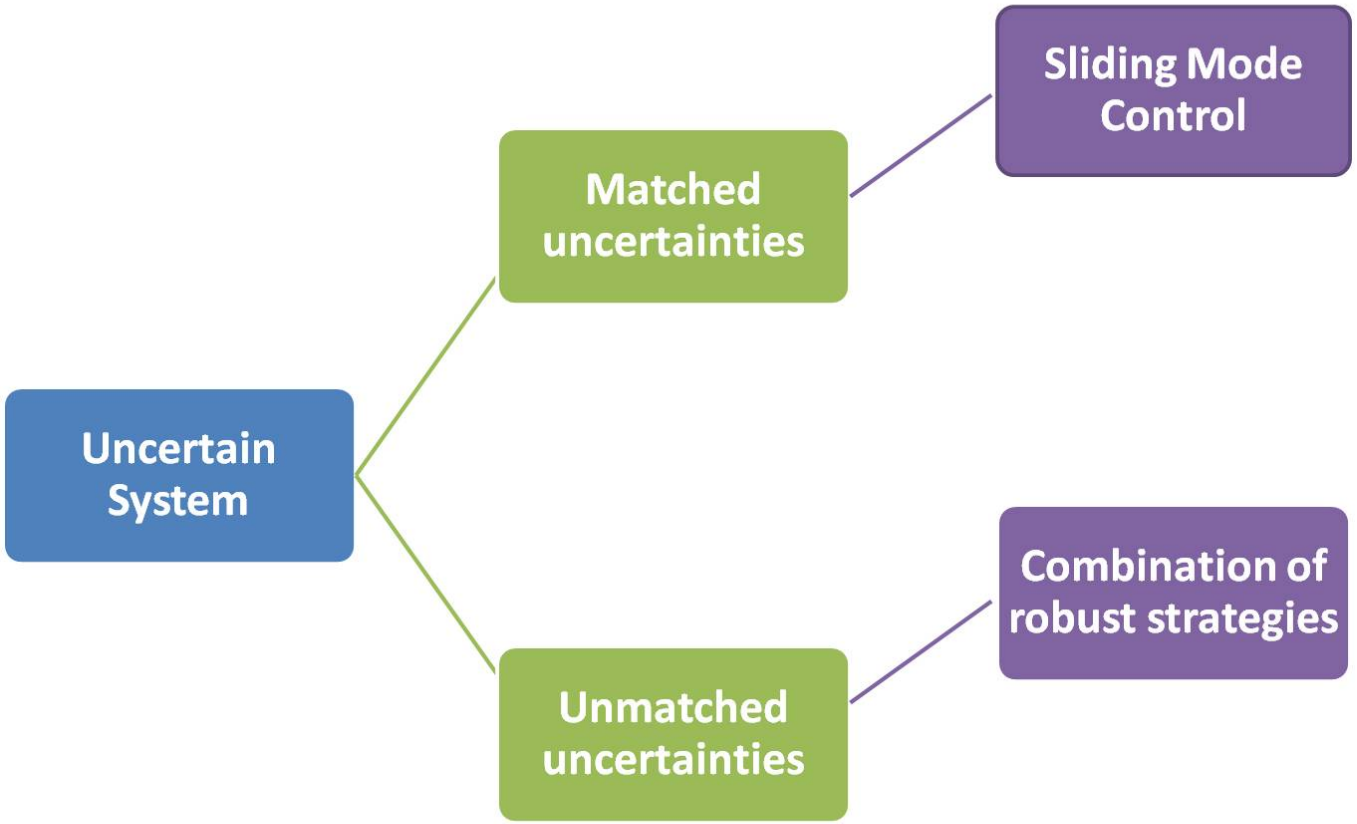
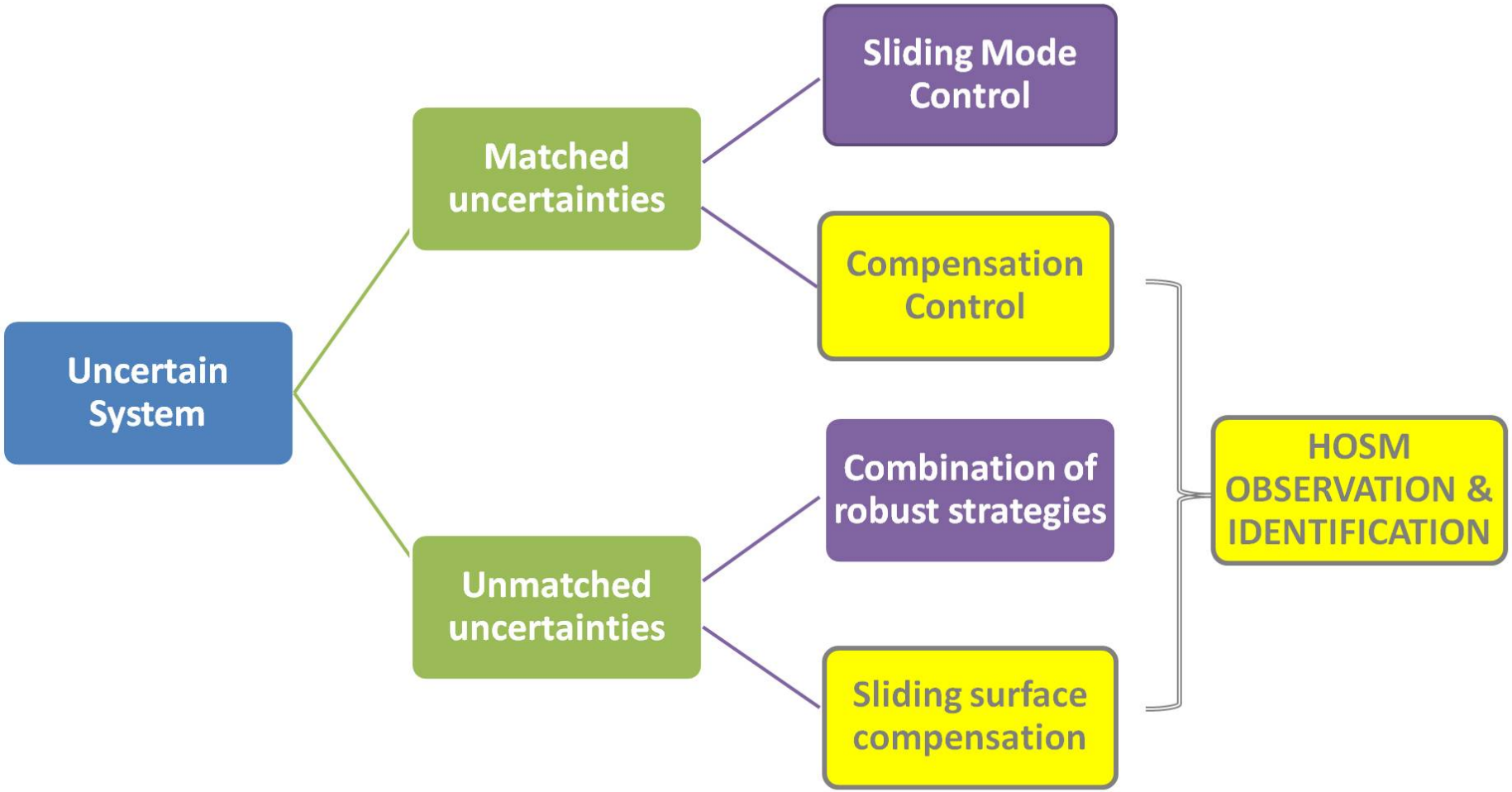
2. Problem statement
3. Methodology: High order sliding mode observer (HOSM)
4. Matched uncertainty compensation
   - Control with or without chattering?
5. Unmatched uncertainty compensation

Motivation


Uncertain System
Matched uncertainties
Unmatched uncertainties
Sliding Mode Control
Combination of robust strategies

Motivation


Uncertain System
Matched uncertainties
Unmatched uncertainties
Sliding Mode Control
Compensation Control
Combination of robust strategies
Sliding surface compensation
HOSM OBSERVATION & IDENTIFICATION

Theoretically exact uncertainty compensation control

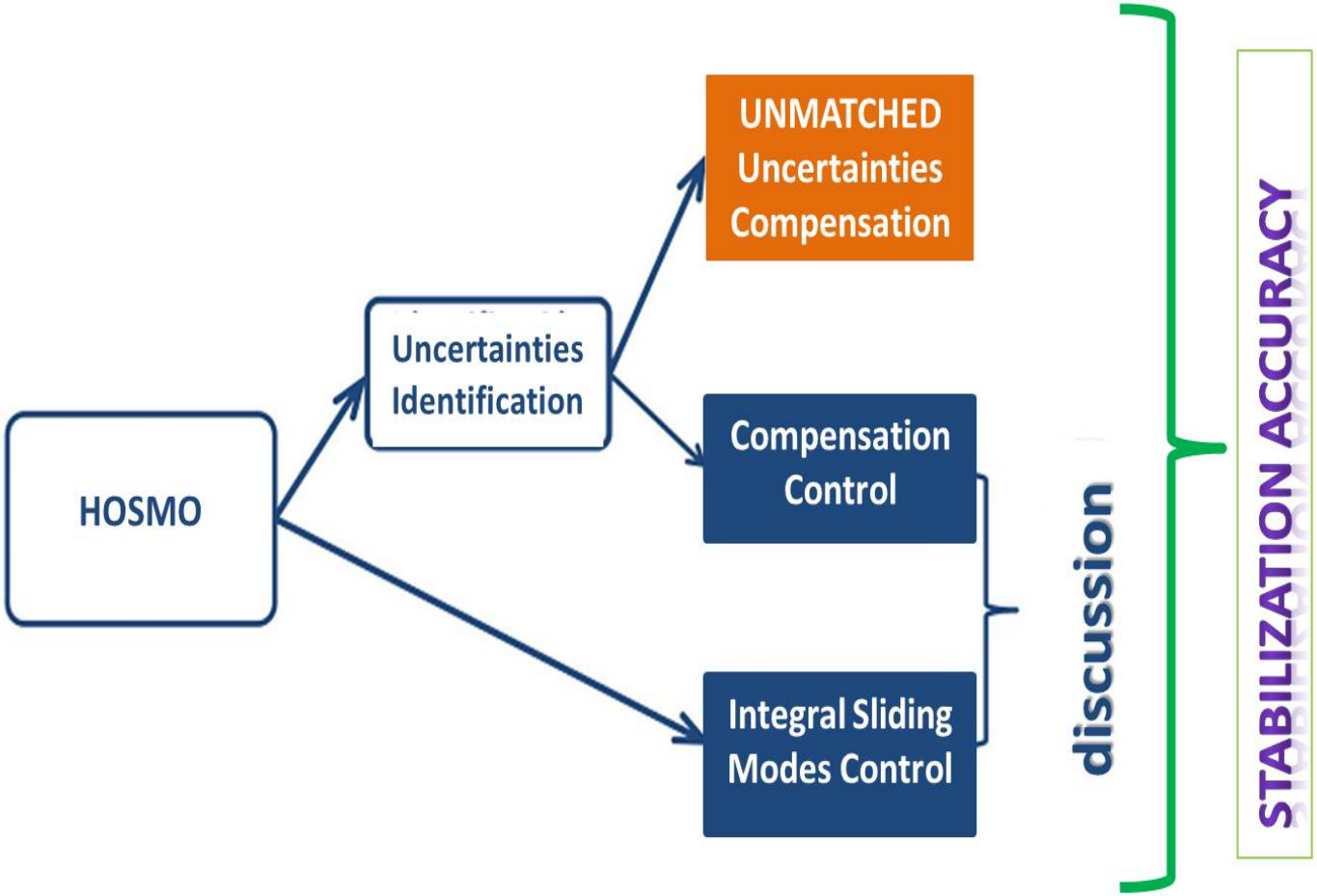
- Sliding mode control
  - Full state measurement is needed
  - Robust on the sliding mode surface
  - *Chattering!!*
  - Robust to matched uncertainties

Starting point . . .

- The advent of high order sliding mode observers
  - Robust in the presence of unknown inputs
  - Theoretically exact state estimation and unknown inputs identification
  - Finite time convergence

- Bejarano, F., Fridman, L., and Poznyak, A., " Exact State Estimation for Linear Systems with Unknown Inputs Based on Hierarchical Super-twisting Algorithm "*International Journal of Robust and Nonlinear Control*, 17, 1734 − 1753.
- J. Dávila, L. Fridman and A. Levant, " Second order sliding mode observer for mechanical systems " , *IEEE Transactions on Automatic Control*, 2006.
- F. Bejarano, L. Fridman, " High order sliding mode observer for linear systems with unbounded unknown inputs," IJC 2010

Contribution



General goals

- Observation and identification based control strategies design for unknown inputs systems (theoretically exact state observation and identification of unknown inputs)
- Accuracy
- Control strategies comparison

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Linear time invariant system with unknown inputs

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) + Dw(t), \\ y(t) &= Cx(t) \end{aligned}$$

$$x(t) \in \Re^n,\ u(t) \in \Re^m,\ y(t) \in \Re^p,\ w(t) \in \Re^q$$

A1. (A, C, D) strongly observable

A2. $w(t)$ bounded with a constantw^+ such that $\|w(t)\| \leq w^+$ for all $t \geq 0$

A3. $w(t)$ satisfies $\left\|w^{(i)}(t)\right\| \leq w^+$ for $i = 1, \ldots, \alpha\ \forall t \geq 0$.

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HOSM algebraic observer

- Luenberger observer (bound the error trajectories)

$$\begin{aligned} \dot{\tilde{x}}(t) &= A\tilde{x}(t) + Bu(t) + L\left(y\left(t\right) - \tilde{y}\left(t\right)\right), \\ \tilde{y}\left(t\right) &= C\tilde{x} \end{aligned}$$

Defining

$$e\left(t\right) := x\left(t\right) - \tilde{x}\left(t\right)$$

- Error dynamics

$$\begin{aligned} \dot{e}\left(t\right) &= \left(A - LC\right)e\left(t\right) + Dw\left(t\right) \\ y_e &= Ce \end{aligned}$$

$$\left\|e\left(t\right)\right\| \leq e^{+}, \text{ for all } t > T$$

Unknown inputs decoupling

Step 1

- $y_e(t) = \underbrace{C}_{M_1} e(t)$

Step 2

- $\frac{d}{dt}\left[(M_1 D)^{\perp} y(t)\right] = (CD)^{\perp} CAx(t)$

$$\left[(M_1 D)^{\perp} M_1\right] \dot{y}(t) = \left[(CD)^{\perp} CA\right] x(t)$$

$$\dot{y}_e = \left[(CD)^{\perp} CA\right] x(t)$$

Step 3

- $\dot{y}_e = \left[(CD)^{\perp} CA\right] \left[M_2 \; M_3\right] \begin{bmatrix} y_e \\ z \end{bmatrix}$

$$\dot{y}_e = \left[(CD)^{\perp} CA M_2 \; \; (CD)^{\perp} CA M_3\right] \begin{bmatrix} y_e \\ z \end{bmatrix}$$

Unknown inputs decoupling

Step 1

- $y_{\mathrm{e}}(t) = \underbrace{C}_{M_1} e(t)$

Step 2

- $\dot{y}_{\mathrm{e}}(t) = M_1 \tilde{A} e(t) + M_1 D w(t)$
- $\underbrace{\begin{bmatrix} \left(\frac{d}{dt} M_1 D\right)^{\perp} M_1 \tilde{A} \\ C \end{bmatrix}}_{M_2} e(t) = \frac{d}{dt} \begin{bmatrix} (M_1 D)^{\perp} & 0 \\ 0 & I_p \end{bmatrix} \begin{bmatrix} y_{\mathrm{e}}(t) \\ \int y_{\mathrm{e}} dt \end{bmatrix}$

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- $\underbrace{\begin{bmatrix} \dfrac{d}{dt}(M_2 D)^{\perp} M_2 \tilde{A} \\ C \end{bmatrix}}_{M_3} e = \frac{d^2}{dt^2}\begin{bmatrix} J_2 & 0 \\ 0 & I_p \end{bmatrix}\begin{bmatrix} y_{\mathrm{e}} \\ \int y_{\mathrm{e}} \\ \int\int y_{\mathrm{e}} d\tau dt \end{bmatrix}$

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State recovering

There exist a $k \leq n$ such that $rank\left(M_k\right) = n$, *(Molinari, 1976)*

Step k

- $$\underbrace{\begin{bmatrix} \frac{d}{dt}\left(M_{k-1}D\right)^{\perp} M_{k-1}\tilde{A} \\ C \end{bmatrix}}_{M_k} e = \frac{d^{k-1}}{dt^{k-1}} \begin{bmatrix} J_{k-1} & 0 \\ 0 & I_p \end{bmatrix} \underbrace{\begin{bmatrix} y_e \\ \int y_e \\ \int \ldots \int y_e d\tau dt \end{bmatrix}}_{Y_{[k-1]}}$$

Then,

$$e\left(t\right) = \frac{d^{k-1}}{dt^{k-1}} \underbrace{M_k^{+} \begin{bmatrix} J_{k-1} & 0 \\ 0 & I_p \end{bmatrix} Y^{[k-1]}}_{\Theta(t)}$$

HOSM differentiator

(Levant,2003)

$$
\begin{aligned}
\dot{z}_0 &= \lambda_0 \Lambda^{\frac{1}{i+1}} \left\| z_0 - \Theta(t) \right\|^{\frac{i}{i+1}} \operatorname{sgn}\left(z_0 - \Theta(t)\right) + z_1 \\
\dot{z}_1 &= \lambda_1 \Lambda^{\frac{1}{i}} \left\| z_1 - \dot{z}_0 \right\|^{\frac{i-1}{i-2}} \operatorname{sgn}\left(z_1 - \dot{z}_0\right) + z_2 \\
&\vdots \\
\dot{z}_{i-1} &= \lambda_{i-1} \Lambda^{\frac{1}{2}} \left\| z_{i-1} - \dot{z}_{i-2} \right\|^{\frac{1}{2}} \operatorname{sgn}\left(z_{i-1} - \dot{z}_{i-2}\right) + z_i \\
\dot{z}_i &= \lambda_i \Lambda \operatorname{sgn}\left(z_i - \dot{z}_{i-1}\right)
\end{aligned}
$$

i: differentiator order, under $A3$ $i = \alpha + \kappa - 1$

$$\frac{d^i \Theta(t)}{dt^i} = z_i \quad \forall t \geq T$$

State estimation

$\forall t \geq T$

$$\hat{x}(t) = \tilde{x}(t) + M_k^{+} z^{(k-1)}$$

Identification of perturbations

From the error dynamics

$$\dot{e}(t) = (A - LC)\, e(t) + Dw(t)$$

Perturbation identification $\forall t \geq T$,

$$\hat{w}(t) = D^{+}(\underbrace{\dot{e}(t)}_{z_k} - (A - LC)\underbrace{e(t)}_{z_{k-1}})$$

Identification of perturbations

From the error dynamics

$$\dot{e}(t) = (A - LC)\, e(t) + Dw(t)$$

Perturbation identification $\forall t \geq T$,

$$\hat{w}(t) = D^{+}(\underbrace{\dot{e}(t)}_{z_k} - (A - LC)\underbrace{e(t)}_{z_{k-1}})$$

Successive derivatives $\forall t \geq T$,

$$\hat{w}^{(i)}(t) = D^{+}(z_{k+i}(t) - (A - LC)\, z_{k+i-1})$$

for all $i = 1, \ldots, \alpha - 1$

Estimation and identification accuracy

Measured output

$$\Theta(t) = \Theta_0(t) + \eta(t)$$

- η *Deterministic noise signal* $\|n(t)\| \leq \eta$
- δ *Sampling step*
- Δ *Combined effect of deterministic noise and sampling time*

(Levant,2003), (Angulo-Levant,2011)

Error	δ	η	Δ
Observation	$O\left(\delta^{\alpha+1}\right)$	$O\left(\nu^{\frac{\alpha+1}{\alpha+k}}\right)$	$O\left(\Delta^{\alpha+1}\right)$
Identification	$O\left(\delta^{\alpha}\right)$	$O\left(\nu^{\frac{\alpha}{\alpha+k}}\right)$	$O\left(\Delta^{\alpha}\right)$
Differentiator	$O\left(\delta\right)$	$O\left(\nu^{\frac{1}{\alpha+k}}\right)$	$O\left(\Delta\right)$

Estimation and identification accuracy

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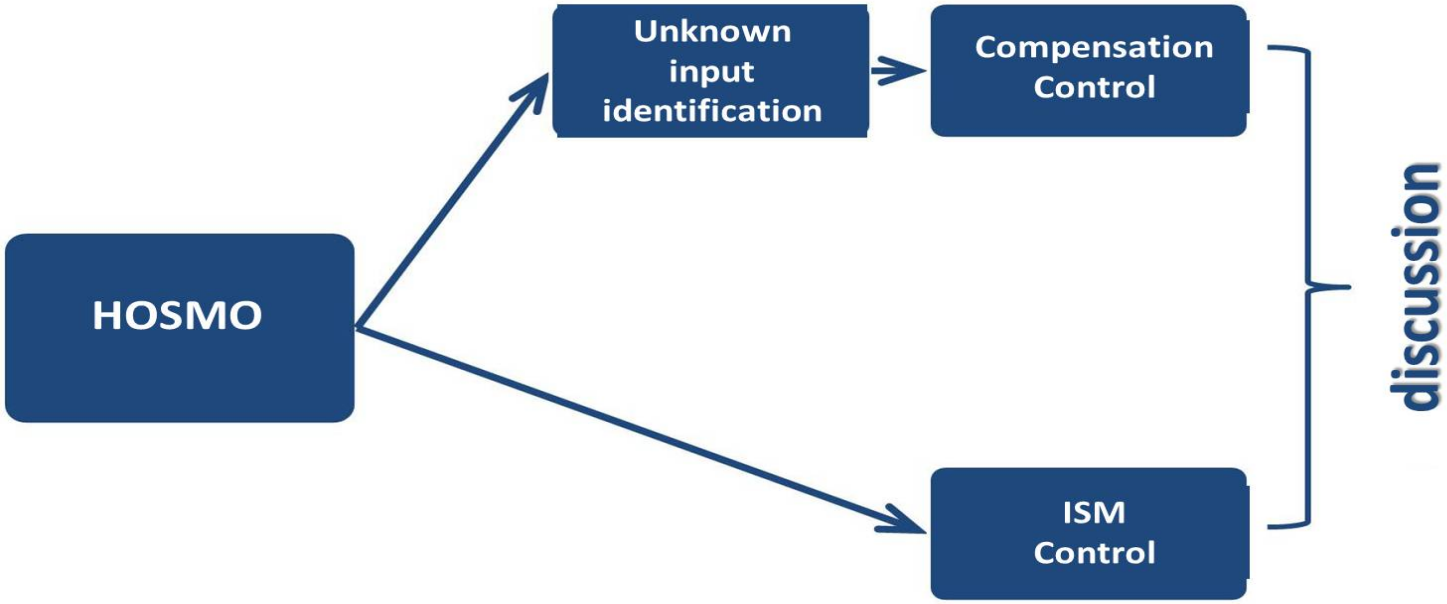
Control with or without chattering

Objectives:

- HOSM based robust control laws design
- Selection criteria for choosing the best control strategy



Control with or without *chattering*

Matched uncertainties system

$$\begin{aligned} \dot{x}(t) &= Ax(t) + B(u(t) + Gw(t)), \\ y(t) &= Cx(t) \end{aligned}$$

Control:

$$u(t) = u_0(t) + u_1(t)$$

Where:

- $u_0(t)$ nominal control, i.e. $(w(t) = 0)$
- $u_1(t)$ uncertainty compensation

Control with or without *chattering*

HOSM compensation control

$$u(t) = -Kx(t) - G\hat{w}(t)$$

Integral sliding mode control *(Utkin-Shi, 1996)*

$$\begin{aligned} u(t) &= -Kx(t) - \rho \frac{s(x,t)}{\|s(x,t)\|} \\ s(x,t) &= B^{+}\left[x(t) - x(T) - \int_{T}^{t} \left[Ax(\tau) + Bu_0(\tau)\right] d\tau\right], \quad \rho > w^{+} \end{aligned}$$

Closed-loop system accuracy ϵ

HOSM compensation control

$$\epsilon = \underbrace{O\left(\Delta^{\alpha+1}\right)}_{Observation} + \underbrace{O\left(\Delta^{\alpha}\right)}_{Identification} + \underbrace{O\left(\mu\right)}_{Execution}$$

Integral sliding mode control

$$\epsilon = \underbrace{O\left(\Delta^{\alpha+1}\right)}_{Observation} + \underbrace{O\left(\mu\right)}_{Execution}$$

Where μ is the actuator time constant with an execution error $O(\mu)$ *(Fridman, 2001, 2002)*.

Discussion...

$$\underbrace{O\left(\Delta^{\alpha}\right)}_{Identification} << \underbrace{O\left(h\right)}_{Execution}$$

Compensation

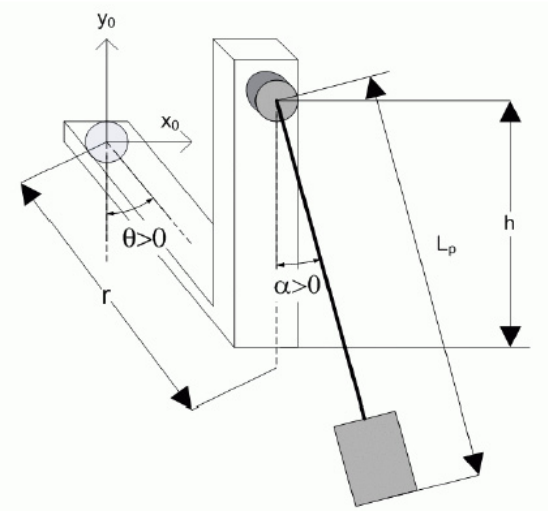
$$\underbrace{O\left(\Delta^{\alpha+1}\right)}_{Observation} << \underbrace{O\left(h\right)}_{Execution} << \underbrace{O\left(\Delta^{\alpha}\right)}_{Identification}$$

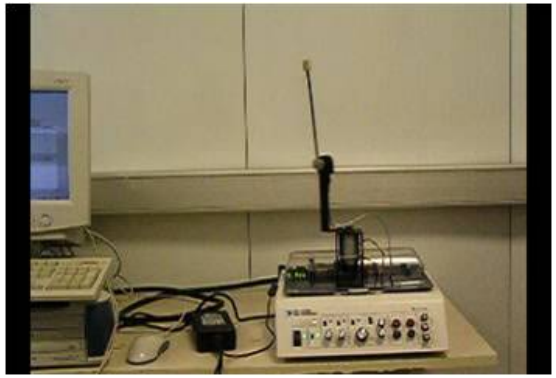
ISM control

$$\underbrace{O\left(h\right)}_{Execution} << \underbrace{O\left(\Delta^{\alpha+1}\right)}_{Observation}$$

Compensation control

Experiments





$$x = \left[\theta, \alpha, \dot{\theta}, \dot{\alpha}\right]^T$$

$$A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 82{,}4 & 1{,}31 & 0 \\ 0 & 56{,}81 & 0{,}37 & 0 \end{bmatrix},$$

$$B = \begin{bmatrix} 0 \\ 0 \\ 46{,}75 \\ 13{,}20 \end{bmatrix}, \qquad D = \begin{bmatrix} 0 \\ 0 \\ 46{,}75 \\ 13{,}20 \end{bmatrix},$$

$$C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

$$w(t) = 0{,}4\mathrm{sen}(2{,}5t) + 0{,}5.$$

Compensation control

ISM Control

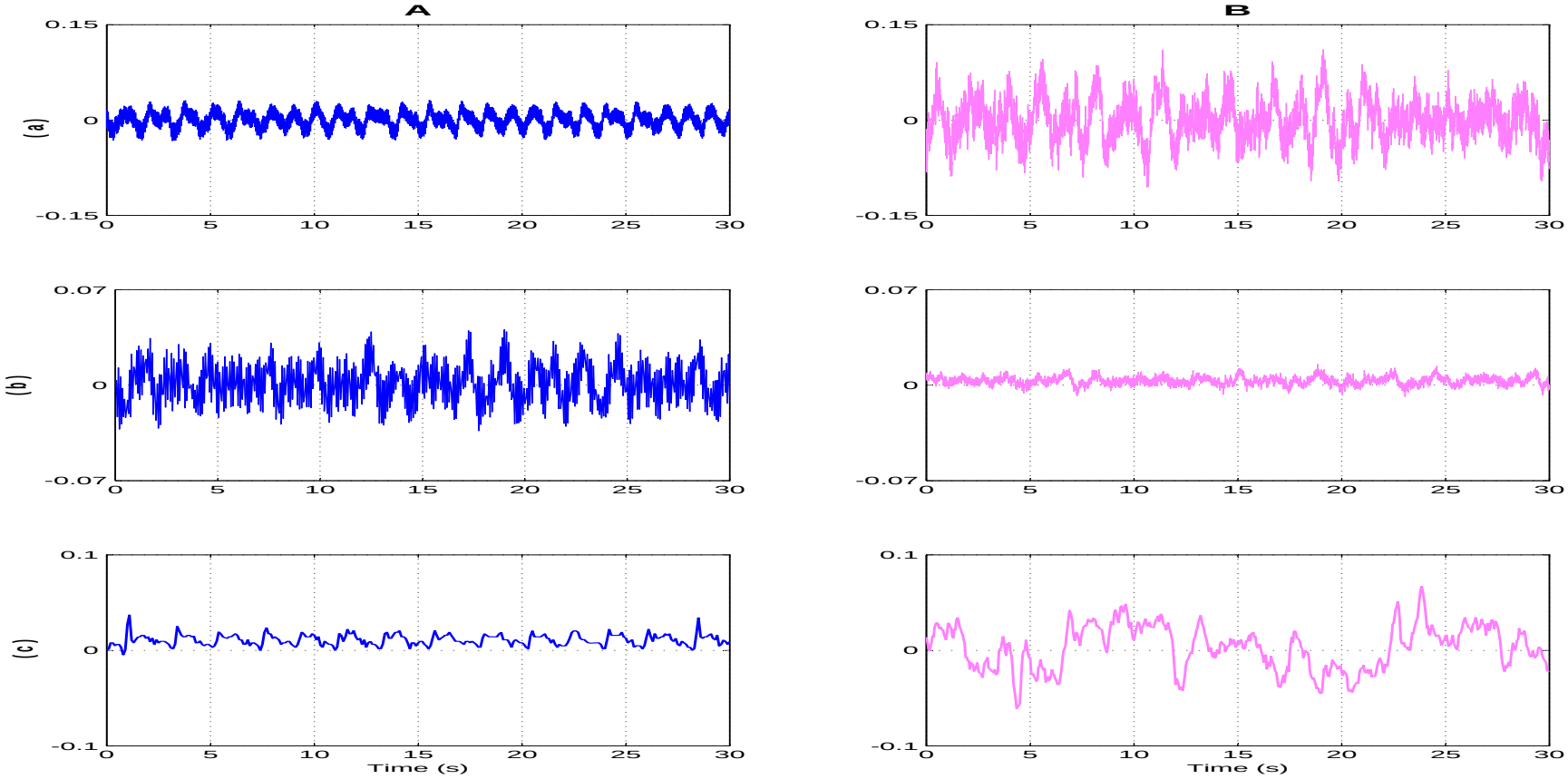


Figura: (a) $\delta = 20(\mu s), \mu = 300(\mu s)$; (b) $\delta = 500(\mu s), \mu = 83(\mu s)$; (c) $\delta = 1000(\mu s), \mu = 83(\mu s)$

ISM

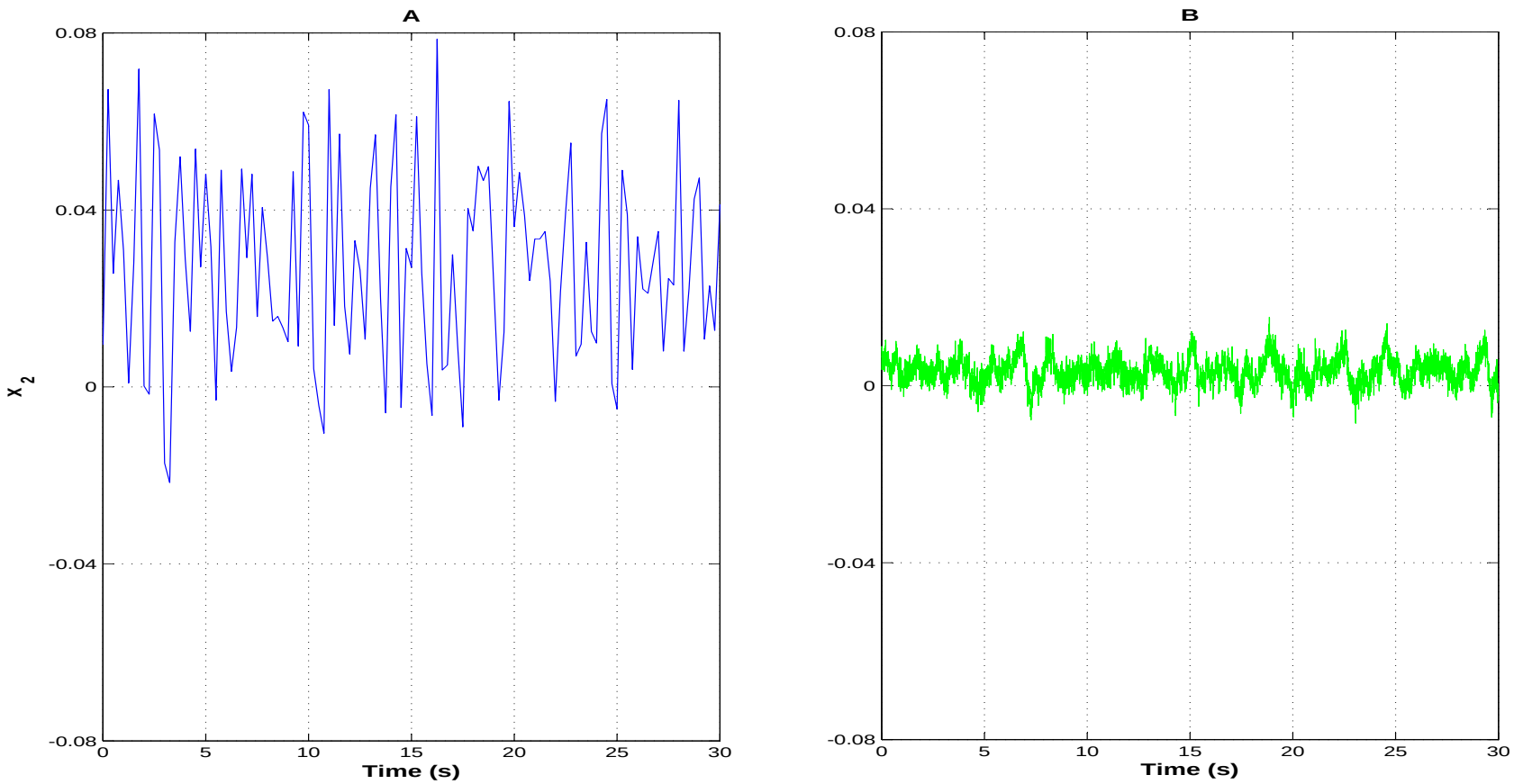


Figura: x_2[rad] using a 1st order HOSM differentiator (izq.) and a 2nd order HOSM differentiator (der).

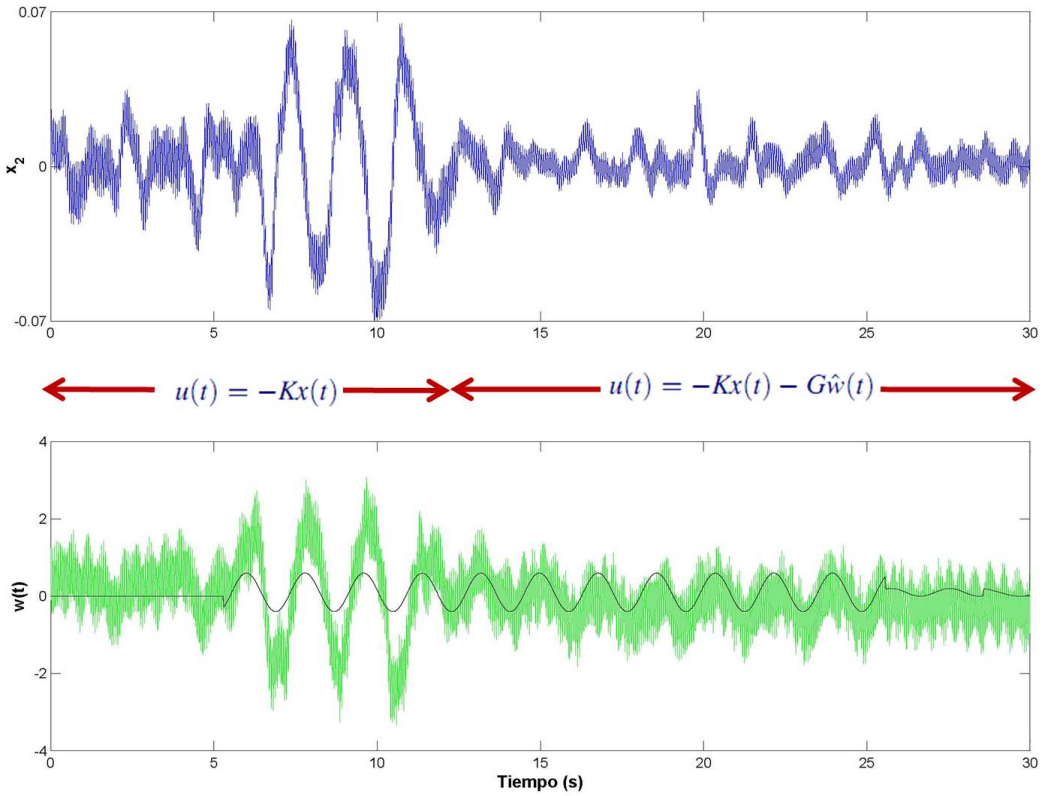


Figura: Pendulum position x_2 [rad] (above). The solid line shows the voltage signal added as a motor perturbation. The total unknown input identified signal $\hat{w}$ is also showed.

Conclusions

- Two robust strategies were proposed:
  - Compensation control based on the estimated state and unknown inputs identification.
  - Integral sliding mode control based on the estimated states.
- A methodology for the selection of an appropriate controller considering the accuracy of observation and identification algorithms as well as the actuator time constant
- The results were validated experimentally in the inverted rotary pendulum system

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Objectives:

- Compensate the effects of the unmatched perturbations and stabilize the subactuated dynamics through the sliding surface
- Maintain the remained states trajectories bounded

System affected by perturbations

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) + Dw(t), \\ y(t) &= Cx(t) \end{aligned} \tag{1}$$

$$x(t) \in \Re^n,\ u(t) \in \Re^m, \text{ and } y(t) \in \Re^p \ (1 \leq p < n)$$

A1. (A, B) controllable

A2. (A, C, D) strongly observable

A3. $w(t)$ and its derivatives up to order r are bounded, i.e. $|\nu| \leq \breve{w}_0$ as well as $|w^{(i)}| \leq \breve{w}_i$ for $i = \overline{1, r}$, for all $t \geq 0$;

Block-controllable form

Transformation into strict feedback form (Loukianov-Utkin,1993),

$$\begin{aligned}\dot{x}_1 &= A_1 x_1 + B_1(x_2 + \Gamma_1 w)\\ \dot{x}_i &= A_i \bar{x}_i + B_i(x_{i+1} + \Gamma_i w)\\ \dot{x}_r &= A_r \bar{x}_r + B_r(u + \Gamma_r w)\end{aligned}$$

$$i = \overline{2, r-1}, \bar{x}_i = [\ x_1^T \quad \dots \quad x_i^T\ ]^T, x_i \in \mathbb{R}^{n_i}, n_i = rank(B_i), \sum_{i=1}^{r} n_i = n$$

- The sub-system comprising $i = \overline{1, r-1}$ represents the sub-actuated dynamics.
- $i = r$ corresponds to the actuated dynamics, $x_r \in \mathbb{R}^m$.

Control goal and methodology

Control aim

The goal is to design an output feedback sliding mode controller u allowing the coordinate x_1 to track a smooth signal x_d in spite of system perturbations w.

- The control relies on the availability, in finite time, of the exact estimation of the state and the identification of the perturbations and their successive derivatives until $r-2$-th order.
- The design of the dynamic sliding surface design takes place in $r-1$-*steps*.
- At $r-th$-*step* the sliding mode control is designed.

Sliding surface design

- ***Step* 1.** The coordinate x_2 can be exploited as a fictitious control input $x_2 := \phi_1$,

$$\phi_1 = -\Gamma_1 \hat{w} - B_1^{\dagger}(A_1 x_1 - \hat{A}_1(x_1 - x_d) - \dot{x}_d)$$

where $\hat{A}_1 \in \mathbb{R}^{n_1}$ is a Hurwitz matrix containing the desired convergence performance of x_1 towards x_d.

$$\vdots$$

- ***Step* i.** The coordinate x_{i+1} is a fictitious control input for (2), it is $x_{i+1} := \phi_i$

$$\phi_i = -\Gamma_i \hat{w} - B_i^{\dagger}(A_i \bar{x}_i - \hat{A}_i(x_i - \phi_{i-1}) + X_{i-1}(x_{i-1} - \phi_{i-2}) - \dot{\phi}_{i-1})$$

Sliding surface design

- ***Step*** r**.** Finally, the sliding surface s is designed like

$$s = x_r - \phi_{r-1}$$

with $s \in \mathbb{R}^m$, $s = \begin{bmatrix} s_1 & \dots & s_m \end{bmatrix}^T$

Control law design

The control law becomes

$$u = -B_r^{\dagger}\left(A_r\bar{x}_r - \dot{\tilde{\phi}}_r - B_{r-1}^T(x_{r-1} - \phi_{r-2}) + \upsilon\right)$$

- with the control $\upsilon \in \mathbb{R}^m$ is the super-twisting control *(Levant,1993)*

$$\upsilon = K_1\Psi_1 + K_2\int_{t_o}^{t}\Psi_2 dt$$

Closed-loop system accuracy ϵ

The control realization error is given by

$$\epsilon = \underbrace{O\left(\Delta^{\alpha+1}\right)}_{Observation} + \underbrace{O\left(\Delta\right)}_{Identification} + \underbrace{O\left(\mu\right)}_{Execution}$$

Example: 3DOF Helicopter

Consider a 3-DOF helicopter

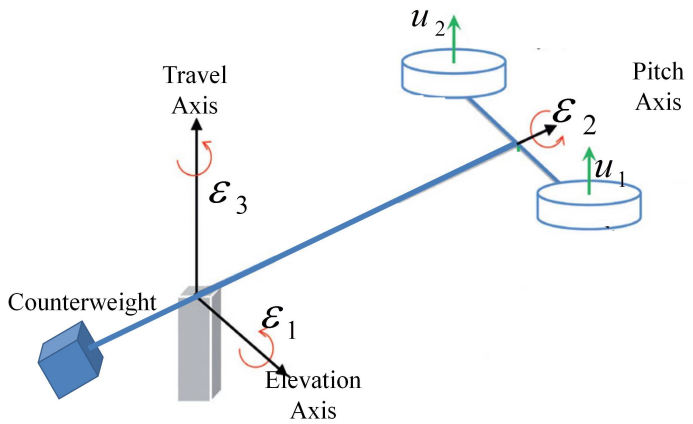

Figura: Schematic diagram of a 3-DOF helicopter.

A linearized model around $\epsilon_2^* = 0$ is given by

$$\begin{aligned}
\ddot{\epsilon}_1 &= 0{,}45\left((u_1 + f_1) + (u_2 + f_2)\right) \\
\ddot{\epsilon}_2 &= 3{,}05\left((u_1 + f_1) - (u_2 + f_2)\right) \\
\ddot{\epsilon}_3 &= -0{,}49\epsilon_2 + \nu
\end{aligned}$$

Example: 3DOF Helicopter

Simulation target

The travel ϵ_3 tracks a desired trajectory $x_d = 5\sin(1{,}1t)$ in spite of faults and disturbances affecting the system.

Simulation trial:

(a) $0 \leq t < 60$ the system is working free of perturbations ($w = 0$, with $w = [\ \nu \quad f_1 \quad f_1\ ]$);

(b) $60 \leq t$ a mismatched disturbance appears $\nu = 2\sin 3t + 1{,}5$;

(c) in $100 \leq t$ a perturbation (type of liquid oscillatory fault) occurs $f_1 = 5\cos(1{,}5t) + 5$;

(d) $140 \leq t$, a perturbation (i.e. drifting fault $f_2 = \beta t : \beta > 0$) appears in the second actuator. The simulation sampling time is $100[\mu s]$.

Simulation results

Sliding Mode Compensation vs H_∞

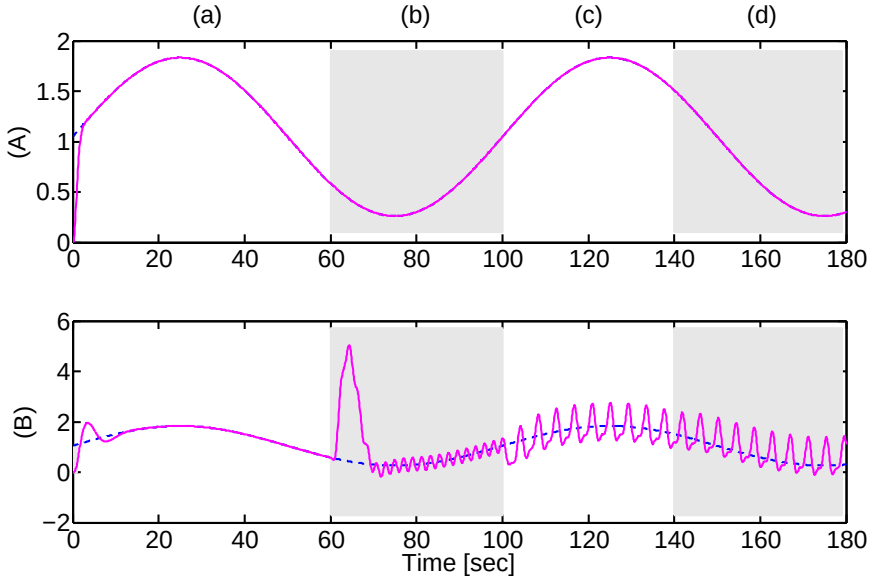

Figura: Travel tracking performance: (A) proposed control and (B) H_∞ control.

Simulation results

Travel performance considering different sampling steps

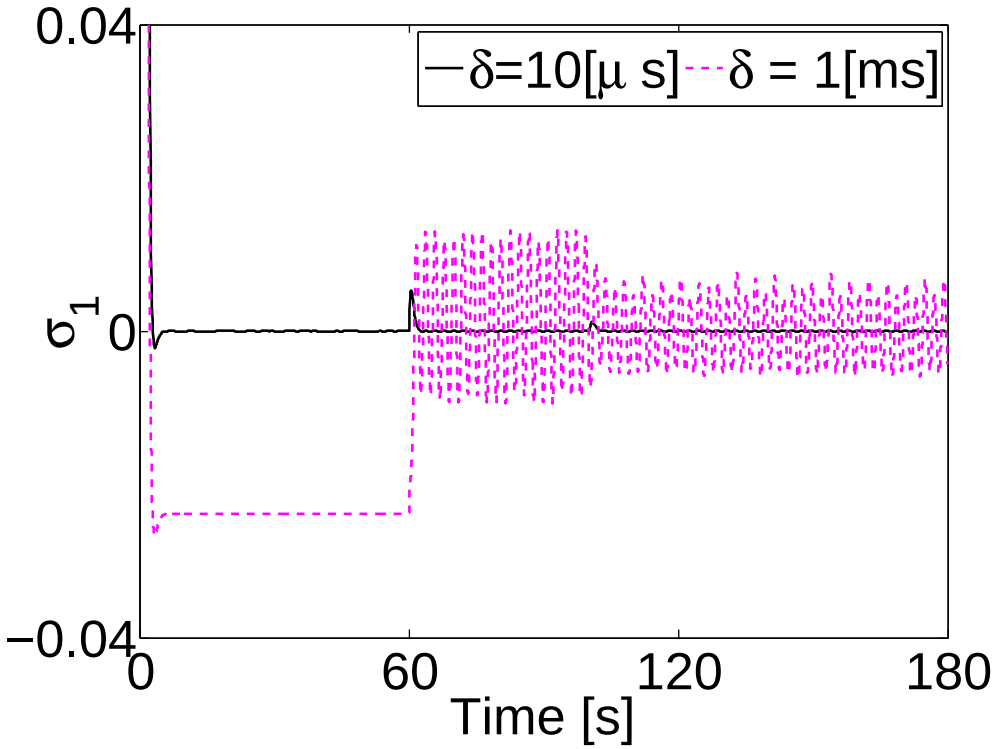

Figura: Tracking error accuracy considering different sampling steps: $\delta = 10[\mu s]$ and $\delta = 1[ms]$.

Conclusions

- A tracking strategy for systems under the effects of unmatched uncertainties was presented.
- Based on the exact reconstruction of the unknown inputs, the effects of the unmatched uncertainties can be compensated through a sliding mode control.
- A nested backward strategy to design a dynamic sliding surface was proposed.
- Simulations in a 3DOF helicopter were done in order to show the feasibility of the strategy.

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Strongly observable system

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Dw(t), \\ y(t) &= Cx(t) \end{aligned}$$

Definition

A system is strongly observable if and only if for any initial condition x_0 and unknown input $w(t)$, the condition $y(t) = 0$ for all $t \geq 0$ implies that $x(t) = 0$ for all $t \geq 0$ *Hautus,1983*.

$$\begin{aligned}\operatorname{rank}\begin{bmatrix} sI - A & B \end{bmatrix} &= \operatorname{rank}\begin{bmatrix} sI - A_1 & -B_1 & 0 \\ -E_1 & sI - F_1 & I_m \end{bmatrix} \\ &= \operatorname{rank}\begin{bmatrix} sI - A_1 & -B_1 \end{bmatrix} + m\end{aligned}$$

$$\operatorname{rank}\begin{bmatrix} sI - A & B \end{bmatrix} = n \Longleftrightarrow \operatorname{rank}\begin{bmatrix} sI - A_1 & -B_1 \end{bmatrix} = n - m$$

$$(A, B) \text{ controllable} \Longleftrightarrow (A_1, B_1) \text{ controllable}$$

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