Sliding Mode Control à la Lyapunov

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Overview

Part I

HOSM Control and Homogeneity

Outline

1. Preliminaries
2. HOSM Control Problem
   - $\rho = 1$, First Order Sliding Mode (FOSM) Control Problem
   - $\rho = 2$, Second Order Sliding Mode (SOSM) Control
3. Homogeneity
   - Classical Homogeneity
   - Weighted Homogeneity
   - Weighted Homogeneity for systems with inputs (perturbations)
   - Weighted Homogeneity and Precision under perturbations
   - Homogeneous Approximation/Domination
   - Example: "Danger" of Non Homogeneous Controllers
4. Homogeneous Design of HOSM (Levant 2005)
5. Plaidoyer for a Lyapunov-Based Framework for HOSM

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Differential Inclusions (DI)

Consider a dynamical system

$$\dot{x} = f(x)\,.$$

We know that

- If $f(x)$ discontinuous according to Filippov we obtain a DI

$$\dot{x} \in F(x)\,,\; F(x) \subset \mathbb{R}^n\,.$$

- If $f(x)$ is uncertain, i.e. $\|f(x)\| \leq f^+$ we can write

$$\dot{x} \in \left[-f^+,\, f^+\right] \Rightarrow \dot{x} \in F(x)\,.$$

- In case of discontinuity or/and uncertainty we obtain a Differential Inclusion from a Differential Equation

Filippov Differential Inclusions (DI)

$$\dot{x} \in F(x)$$

is Filippov DI if $\forall x \in \mathbb{R}^n$, the set-valued function $F(x) \subset \mathbb{R}^n$ is

- not empty;
- compact;
- convex;
- upper-semicontinuous, i.e.

$$\lim_{y \to x} \sup \left[\{ \mathrm{dist}(z, F(x)) | z \in F(y) \} \right] = 0$$

  where

$$\mathrm{dist}(x,\ A) = \inf \left\{ |x - a| | a \in A \right\} .$$

- A solution $x(t)$ is an absolutely continuous function satisfying the DI almost everywhere.
- Filippov DI have the usual properties, except for uniqueness.

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Higher Order Sliding Mode (HOSM)

Consider a Filippov DI $\dot{x} \in F(x)$, with a smooth output function $\sigma = \sigma(x)$. If

1. The total time derivatives $\sigma, \dot{\sigma}, \ldots, \sigma^{(r-1)}$ are continuous functions of x
2. The set
$$\sigma = \dot{\sigma} = \ldots = \sigma^{(r-1)} = 0 \tag{1}$$
is a nonempty integral set (i.e., consists of Filippov trajectories)
3. The Filippov set of admissible velocities at the r-sliding points (1) contains more than one vector

the motion on the set (1) is said to exist in an r -sliding (rth-order sliding) mode. The set (1) is called the r-sliding set. The nonautonomous case is reduced to the one considered above by introducing the fictitious equation $\dot{t} = 1$.

Overview

HOSM Control

SISO smooth, uncertain system

$$\dot{z} = f(t, z) + g(t, z)\, u, \;\; \sigma = h(t, z),$$

- $z \in \mathbb{R}^n$, $u \in \mathbb{R}$, $\sigma \in \mathbb{R}$: sliding variable.
- $f(t, z)$ and $g(t, z)$ and n uncertain.
- Control objective: to reach and keep $\sigma \equiv 0$ in finite time.
- Relative Degree ρ w.r.t. σ is well defined, known and constant.
- Reduced (Zero) Dynamics asymptotically stable (by appropriate selection of σ).

The basic DI

Defining $x = (x_1, \ldots, x_\rho)^T = (\sigma, \dot{\sigma}, \ldots, \sigma^{(\rho-1)})^T$, $\sigma^{(i)} = \frac{d^i}{dt^i} h(z,t)$

The regular form

$$\sum\nolimits_T : \begin{cases} \dot{x}_i = x_{i+1}, & i = 1, \ldots, \rho - 1, \\ \dot{x}_\rho = w(t, z) + b(t, z)\, u, & x_0 = x(0)\ , \\ \dot{\zeta} = \phi(\zeta, x) & \zeta_0 = \zeta(0)\,, \end{cases}$$

$$0 < K_m \le b(t, z) \le K_M,\ |w(t, z)| \le C\,.$$

Reduced Dynamics Asymptotically stable:

$$\dot{\zeta} = \phi(\zeta, 0)\, \zeta_0 = \zeta(0)\,,$$

The basic Differential Inclusion (DI)

$$\sum\nolimits_{DI} : \begin{cases} \dot{x}_i = x_{i+1},\ i = 1, \ldots, \rho - 1, \\ \dot{x}_\rho \in [-C, C] + [K_m, K_M] u\,. \end{cases}$$

The Basic Problems

Bounded memoryless feedback controller

$$u = \vartheta_\rho(x_1, x_2, \cdots, x_\rho),$$

- Render $x_1 = x_2 = \cdots = x_\rho = 0$ finite-time stable.
- Motion on the set $x = 0$ is *ρth-order sliding mode.*
- ϑ_ρ necessarily discontinuos at $x = 0$ for robustness $[-C, C]$.

Problem 1

How to design an appropriate control law ϑ_r ?

Problem 2

How to estimate in finite time the required derivatives
$x = (x_1, ..., x_\rho)^T = (\sigma, \dot{\sigma}, ..., \sigma^{(\rho-1)})^T$?

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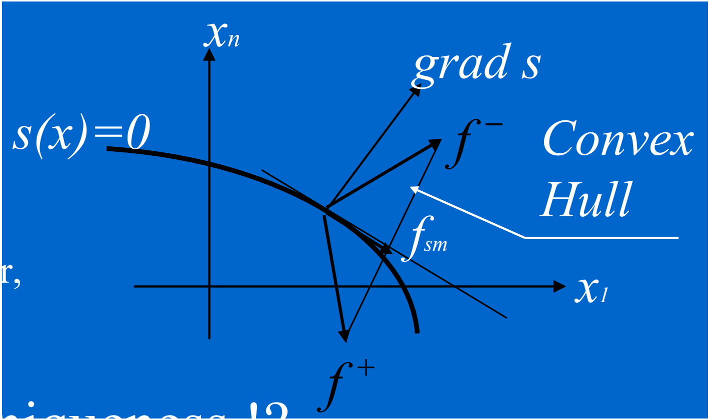
$\rho = 1$, First Order Sliding Mode (FOSM) Control

- Introduced in the mid 60's: Fantastic results, mature theory, lots of applications,... Utkin, Emelianov, ... Edwards, Spurgeon, Sira-Ramirez, Loukianov, Zinober, ...
- $\dot{\sigma} \in [-C, C] + [K_m, K_M]u$.
- $u = \vartheta_1 = -k \operatorname{sign}(\sigma)$, $k > \frac{C}{K_m}$.
- Robust (=insensitive) control, simple realization, finite time convergence to the sliding manifold,...
- Lyapunov design using $V(\sigma) = \frac{1}{2}\sigma^2$.
- No derivative estimation required.
- Mature theory including Multivariable case, adaptation, design of sliding surfaces, ...



x_n
grad s
$s(x)=0$
f^-
Convex
Hull
f_{sm}
x_1
f^+
r,

Continuous vs. Discontinuous Control: A first order plant

Consider a plant

$$\dot{\sigma} = \alpha + u, \ \ \sigma(0) = 1$$

where $\alpha \in (-1, 1)$ is a perturbation.
Continuous (linear) Control

$$\dot{\sigma} = \alpha - k\sigma, \ \ k > 0, \ \ \sigma(0) = 1$$

Comments:

- RHS of DE continuous (linear).
- If $\alpha = 0$ exponential (asymptotic) convergence to $\sigma = 0$.
- If $\alpha \neq 0$ practical convergence.

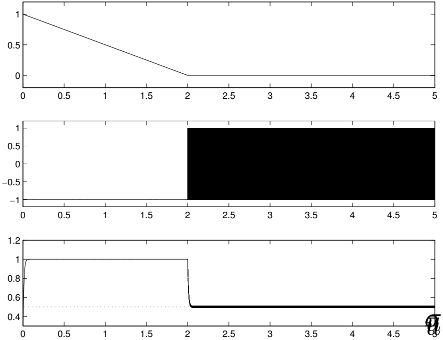
Discontinuous Control

$$\dot{\sigma} = \alpha - \text{sign}(\sigma), \quad \sigma(0) = 1$$

with $\alpha \in (-1, 1)$.

- $\sigma > 0 \Rightarrow \dot{\sigma} =< 0$
- $\sigma < 0 \Rightarrow \dot{\sigma} => 0$

y $\sigma(t) \equiv 0, \forall t \geq T$.

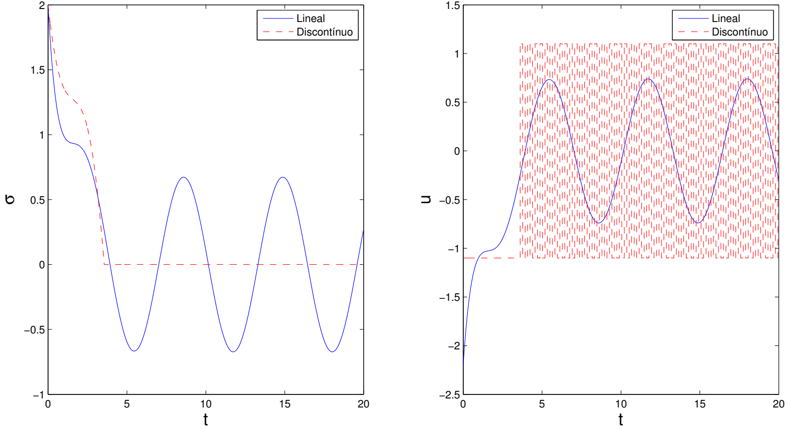


Comments:

- ¿$0 = \alpha - \text{sign}(0)$?
- RHS of DE is discontinuous.
- After arriving at $\sigma = 0$, sliding on $\sigma \equiv 0$.

- Finite-Time convergence.
- Differential Inclusion.

$$\dot{\sigma} \in [-\alpha, \alpha] - \text{sign}(\sigma)$$



Notice the chattering = infinite switching of the control variable!

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$\rho = 2$, Second Order SM (SOSM) Control

- Introduced in the mid 80's: Wonderful results, mature geometric theory, lots of applications,... Levant, Fridman, Bartolini, Ferrara, Shtessel, Usai, Feng, Man, Yu, Furuta, Spurgeon, Orlov, Perruquetti, Barbot, Floquet, Defoort, ...
- $$\sum_{DI} : \begin{cases} \dot{x}_1 = x_2, \\ \dot{x}_2 \in [-C, C] + [K_m, K_M]u \, . \end{cases}$$
- Some controllers (see Fridman 2011):
  - Twisting Controller (Emelyanov, Korovin, Levant 1986). $\vartheta_2(x) = -k_1 \mathrm{sign}(x_1) - k_2 \mathrm{sign}(x_2)$.
  - Super-Twisting Algorithm (Levant 1993) (as differentiator Levant 1998).
  - The Sub-Optimal Algorithm (Bartolini, Ferrara, Usai 1997).
  - Terminal Sliding Mode Control (Man, Paplinski, Wu, Yu (1994, 1997..)).

A second order plant

$$
\begin{aligned}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= \phi(x_1, x_2) + u \\
y &= x_1
\end{aligned}
$$

ϕ: Perturbation/uncertainty.

Question: Can we just feedback the output (as for FO case)?

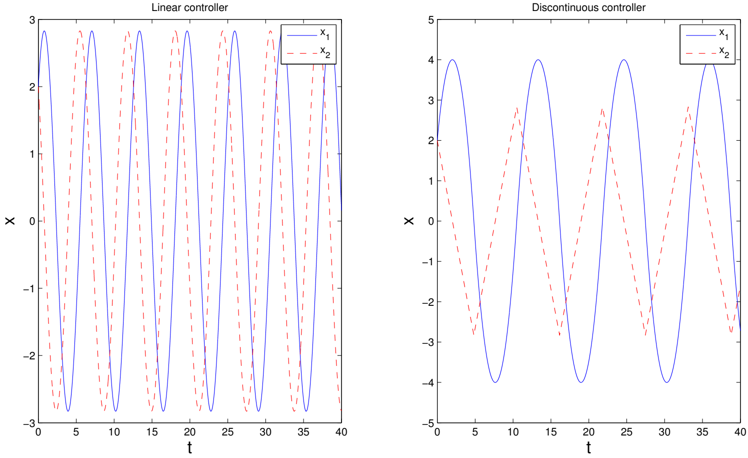
Two alternative *output* controllers:

- Continuous (linear) output controller (Homogeneous Time Invariant (HTI))

$$
\begin{aligned}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= -k_1 y
\end{aligned}
$$

- Discontinuous output controller (HTI)

$$
\begin{aligned}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= -k_1 \text{sign}(y)
\end{aligned}
$$



Both Oscillate!

It is impossible to stabilize a double (or triple etc) integrator by static output feedback!

State feedback

State feedback controllers:

- Continuous (linear) state feedback controller (HTI)

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -k_1 x_1 - k_2 x_2 \end{aligned}$$

  - Exponential Convergence
  - Robust, but Sensitive to perturbations: Practical stability.
- Continuous HTI state feedback controller

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -k_1 \lceil x_1 \rfloor^{\frac{1}{3}} - k_2 \lceil x_2 \rfloor^{\frac{1}{2}} \end{aligned}$$

$\lceil \cdot \rfloor^{\rho} = |\cdot|^{\rho} \operatorname{sign}(\cdot)$

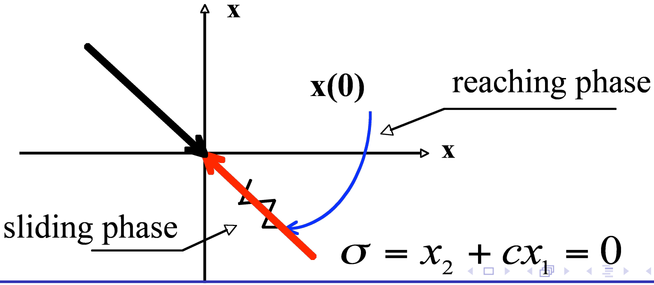
  - Finite Time Convergence
  - Robust, but Sensitive to perturbations: Practical stability.

First Order Sliding Mode Controller

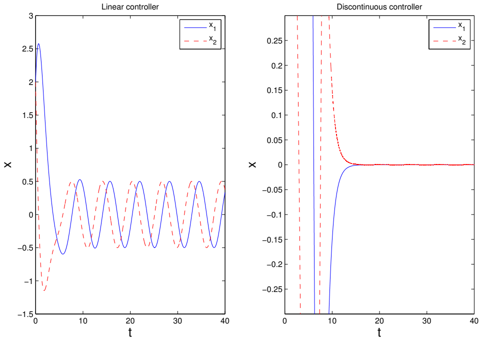
$$
\begin{aligned}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= -k_2 \text{sign}(x_2 + k_1 x_1)
\end{aligned}
$$

Rewrite as a first order system with a stable (first order) zero dynamics: with $\sigma = x_2 + k_1 x_1$ sliding variable

$$
\begin{aligned}
\dot{x}_1 &= -k_1 x_1 + \sigma \\
\dot{\sigma} &= -k_2 \text{sign}(\sigma)
\end{aligned}
$$



Behavior with perturbation

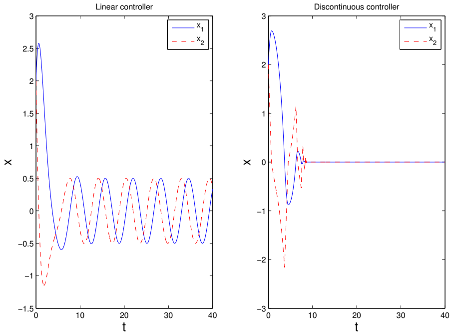


- Linear controller stabilize exponentially and is not insensitive to perturbation
- SM control also stabilizes exponentially but is insensitive to perturbation!

The Twisting Controller

A discontinuous HTI controller able to obtain finite-time convergence and insensitivity to perturbations:

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -k_1 \mathrm{sign}(x_1) - k_2 \mathrm{sign}(x_2) \end{aligned}$$



$\rho = 2$, SOSM Control....

- Robust (=insensitive) control, finite time convergence to the sliding manifold,...
- No Lyapunov design or analysis.
- Analysis and Design is very geometric: Beautiful but difficult to extend to $\rho > 2$.
- Solution: Homogeneity!

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Classical Homogeneity

Classical Homogeneity for functions. (Euler, Zubov, Hahn,...)

Let n, m be positive integers. A mapping $f : \mathbb{R}^n \to \mathbb{R}^m$ is homogeneous with *degree* $\delta \in \mathbb{R}$ iff $\forall \lambda > 0$:

$$f(\lambda x) = \lambda^{\delta} f(x).$$

Some examples:

- Linear function: Let $A \in \mathbb{R}^{m \times n}$ then $f(x) = Ax$ is homogeneous of degree $\delta = 1$, since

$$f(\lambda x) = A(\lambda x) = \lambda (Ax) = \lambda f(x) .$$

- $f(x_1, x_2) = \frac{x_1^3 + x_2^3}{x_1^2 + x_2^2}$ is continuous and homogeneous of degree $\delta = 1$ (but not linear!), since

$$f(\lambda x_1, \lambda x_2) = \frac{(\lambda x_1)^3 + (\lambda x_2)^3}{(\lambda x_1)^2 + (\lambda x_2)^2} = \frac{\lambda^3 (x_1^3 + x_2^3)}{\lambda^2 (x_1^2 + x_2^2)} = \lambda f(x_1, x_2) .$$

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- $$f(x_1,\, x_2) = \begin{cases} \frac{\lceil x_1 \rfloor^{\frac{1}{2}} + \lceil x_2 \rfloor^{\frac{1}{2}}}{x_1 + x_2} & \text{if } x_1 + x_2 \neq 0 \\ 0 & \text{otherwise} \end{cases}$$

  is discontinuous and homogeneous of degree $\delta = -\frac{1}{2}$, since

  $$f(\lambda x_1,\, \lambda x_2) = \frac{\lceil \lambda x_1 \rfloor^{\frac{1}{2}} + \lceil \lambda x_2 \rfloor^{\frac{1}{2}}}{\lambda x_1 + \lambda x_2} = \lambda^{-\frac{1}{2}} f(\lambda x_1,\, \lambda x_2)$$

- Quadratic Form: if $x \in \mathbb{R}^n$ and $P \in \mathbb{R}^{n \times n}$, $q(x) = x^T P x$ is homogeneous of degree $\delta = 2$, since

  $$q(\lambda x) = (\lambda x)^T P (\lambda x) = \lambda^2 x^T P x = \lambda^2 q(x)\,.$$

- Classical Form = homogeneous polynomial: if $x \in \mathbb{R}^n$, e.g.

  $$p(x) = \alpha_1 x_1 x_2 x_3 + \alpha_2 x_1 x_3 x_5 + \alpha_3 x_1^2 x_5 + \alpha_4 x_2^3 + \cdots$$

  is homogeneous of degree $\delta = 3$.

-
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  is homogeneous of degree $\delta = 3$.

Classical Homogeneity

Classical Homogeneity for vector fields. (Zubov, Hahn,...)

Let n be a positive integer. A vector field $f : \mathbb{R}^n \to \mathbb{R}^n$ is homogeneous with *degree* $\delta \in \mathbb{R}$ iff $\forall \lambda > 0$:

$$f(\lambda x) = \lambda^{\delta} f(x).$$

Associated with the vector field $f(x)$ is the Differential Equation $\dot{x} = f(x)$, and it has a flow (solution) $\varphi(t, x)$.

Homogeneity of vector field $\Rightarrow$ Homogeneity of Flow

If $\forall \lambda > 0$:

$$f(\lambda x) = \lambda^{\delta} f(x) \Rightarrow \varphi(t, \lambda x) = \lambda \varphi(\lambda^{\delta-1} t, x)$$

Classical Homogeneity

Classical Homogeneity for vector fields. (Zubov, Hahn,...)

Let n be a positive integer. A vector field $f : \mathbb{R}^n \to \mathbb{R}^n$ is homogeneous with *degree* $\delta \in \mathbb{R}$ iff $\forall \lambda > 0$:

$$f(\lambda x) = \lambda^{\delta} f(x).$$

Associated with the vector field $f(x)$ is the Differential Equation $\dot{x} = f(x)$, and it has a flow (solution) $\varphi(t, x)$.

Homogeneity of vector field $\Rightarrow$ Homogeneity of Flow

If $\forall \lambda > 0$:

$$f(\lambda x) = \lambda^{\delta} f(x) \Rightarrow \varphi(t, \lambda x) = \lambda \varphi(\lambda^{\delta - 1} t, x)$$

Some examples:

- Linear system: Let $A \in \mathbb{R}^{n\times n}$ then $\dot{x} = f(x) = Ax$ is homogeneous of degree $\delta = 1$ and the flow is $\varphi(t, x) = e^{At}x$

$$\varphi(t, \lambda x) = e^{At}(\lambda x) = \lambda e^{At}x = \lambda\varphi(\lambda^0 t, x).$$

- If x is scalar. System $\dot{x} = -\text{sign}(x)$ is homogeneous with degree $\delta = 0$, since

$$f(\lambda x) = \text{sign}(\lambda x) = \text{sign}(x) = \lambda^0 f(x).$$

The flow is

$$\varphi(t, x) = \begin{cases} \text{sign}(x)(|\,x\,| - t) & \text{if } 0 \leq t \leq |\,x\,| \\ 0 & \text{if } t > |\,x\,| \end{cases}$$

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Outline

Weighted Homogeneity or (quasi-homogeneity) (Zubov, Hermes)

- A generalized weight is a vector $\mathbf{r} = (r_1, \cdots, r_n)$, with $r_i > 0$.
- A dilation is the action of the group $\mathbb{R}_+ \setminus \{0\}$ on $\mathbb{R}^n$ given by

$$\begin{aligned} \Lambda_{\mathbf{r}} : \mathbb{R}_+ \setminus \{0\} \times \mathbb{R}^n &\rightarrow \mathbb{R}^n \\ (\lambda, x) &\rightarrow \mathrm{diag}\{\lambda^{r_i}\} x \end{aligned}$$

we will denote this for simplicity as $\Lambda_{\mathbf{r}} x \triangleq \Lambda_{\mathbf{r}}(\lambda, x)$, $\lambda > 0$.

Weighted Homogeneity for functions. (Zubov, Hermes...)

Let n, m be positive integers. A mapping $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$ is $\mathbf{r}$-homogeneous with *degree* $\delta \in \mathbb{R}$ iff $\forall \lambda > 0$:

$$f(\Lambda_{\mathbf{r}} x) = \lambda^{\delta} f(x).$$

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Some remarks

- Function $f(x_1, x_2) = x_1 + x_2^2$ is not homogeneous, but it is $(2, 1)$-homogeneous of degree $\delta = 2$ since,

$$f(\lambda x) = \lambda x_1 + (\lambda x_2)^2 = \lambda x_1 + \lambda^2 x_2^2 \neq \lambda^\delta f(x),\ \forall \lambda \neq 1\,.$$

$$f(\Lambda_{\mathbf{r}} x) = \lambda^2 x_1 + (\lambda x_2)^2 = \lambda^2 (x_1 + x_2^2) = \lambda^2 f(x)\,.$$

- Classical Homogeneity = Weighted Homogeneity with $\mathbf{r} = (r_1, \cdots, r_n) = (1, \cdots, 1)$.
- Values on the unit sphere define an $\mathbf{r}$-Homogeneous function.
- If $f(x)$ is $\mathbf{r}$-homogeneous of degree δ then it is $(\alpha \mathbf{r})$-homogeneous of degree $(\alpha\delta)$ for any $\alpha > 0$
- Euler's Theorem: Let $V : \mathbb{R}^n \to \mathbb{R}$ be differentiable. V is $\mathbf{r}$-homogeneous of degree δ if and only if

$$\sum_{i=1}^{n} r_i x_i \frac{\partial V}{\partial x_i}(x) = \delta V(x)\,.$$

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Weighted Homogeneity for (set-valued) vector fields. (Zubov, Hermes, Levant, ...)

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Associated with the (set-valued) vector field $f(x)$ is the Differential Equation $\dot{x} = f(x)$ (DI $\dot{x} \in f(x)$), and it has a flow (solution) $\varphi(t, x)$.

r-Homogeneity of vector field $\Rightarrow$ **r**-Homogeneity of Flow

If $\forall \lambda > 0$:

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Some examples:

- Super-Twisting (ST) Algorithm:

$$\begin{aligned}\dot{x}_1 &= -k_1 \lceil x_1 \rfloor^{\frac{1}{2}} + x_2 \\ \dot{x}_2 &\in -k_2 \lceil x_1 \rfloor^{0} + [-1,\, 1]\,.\end{aligned}$$

  is $(2,1)$-homogeneous of degree -1, since

$$\begin{aligned}-k_1 \lceil \lambda^2 x_1 \rfloor^{\frac{1}{2}} + \lambda x_2 &= \lambda^{2-1}(-k_1 \lceil x_1 \rfloor^{\frac{1}{2}} + x_2) \\ -k_2 \lceil \lambda^2 x_1 \rfloor^{0} + [-1,\, 1] &= \lambda^{1-1}(-k_2 \lceil x_1 \rfloor^{0} + [-1,\, 1])\,.\end{aligned}$$

- Twisting Algorithm:

$$\begin{aligned}\dot{x}_1 &= x_2 \\ \dot{x}_2 &\in -k_1 \lceil x_1 \rfloor^{0} - k_2 \lceil x_2 \rfloor^{0} + [-1,\, 1]\,.\end{aligned}$$

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Two important examples:

- Levant's arbitrary order differentiator:

$$\begin{aligned}\dot{x}_i &= -k_i \lceil x_1 - f(t) \rfloor^{\frac{n-i}{n}} + x_{i+1} \\ &\vdots \\ \dot{x}_n &\in -k_n \lceil x_1 - f(t) \rfloor^{0} .\end{aligned}$$

  is $(n, n-1, \cdots, 1)$-homogeneous of degree -1.

- Homogeneous Controller of a chain of integrators:

$$\begin{aligned}\dot{x}_1 &= x_2 \\ &\vdots \\ \dot{x}_n &= -\sum_{i=1}^{n} k_i \lceil x_i \rfloor^{\alpha_i}\end{aligned}$$

  is $\mathbf{r}$-homogeneous of degree $\delta \in [-1, 0]$ with

$$r_i = 1 + (i-n)\delta, \ \alpha_i = \frac{1+\delta}{1+(i-n)\delta} .$$

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Dynamic interpretation of r-homogeneity

- System $\dot{x} = f(x)$ is $\mathbf{r}$-homogeneous of degree δ, i.e. $f(\Lambda_{\mathbf{r}} x) = \lambda^{\delta} \Lambda_{\mathbf{r}} f(x)$.
- State Transformation $z = \Lambda_{\mathbf{r}} x$

$$\dot{z} = \Lambda_{\mathbf{r}} \dot{x} = \Lambda_{\mathbf{r}} f(x) = \lambda^{-\delta} f(\Lambda_{\mathbf{r}} x)$$

and therefore

$$\frac{dz}{d(\lambda^{\delta} t)} = f(z)$$

- System $\dot{x} = f(x)$ is invariant under the transformation

$$G_{\lambda} : (t, x) \mapsto \left(\lambda^{-\delta} t, \Lambda_{\mathbf{r}} x\right).$$

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Properties of Homogeneous Systems

- Zubov, Hahn, Hermes, Kawski, Rosier, Aeyels, Sepulchre, Grüne, Praly, Perruquetti, Efimov, Polyakov,..... Levant, Orlov, Bernuau et al. 2013
- If $x = 0$ Locally Attractive (LA) $\Leftrightarrow$ Globally Asymptotically Stable (GAS)
- local contraction $\Rightarrow$ global contraction $\Rightarrow$ global asymptotic stability
- If $x = 0$ GAS and $\delta < 0 \Leftrightarrow x = 0$ Finite Time Stable
- If $x = 0$ GAS and $\delta = 0 \Leftrightarrow x = 0$ Exponentially Stable (e.g. LTI systems)
- If $x = 0$ GAS and $\delta > 0 \Leftrightarrow x = 0$ Asymptotically Stable
- If $x = 0$ GAS $\Leftrightarrow$ It exists a Homogeneous Lyapunov Function

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Outline

Weighted Homogeneity with inputs

Consider a System with inputs $u \in \mathbb{R}^m$

$$\dot{x} = f(x, u) .$$

- State and input weight vectors $\mathbf{r} = (r_1, \cdots, r_n)$, $r_i > 0$, $\rho = (\rho_1, \cdots, \rho_m)$, $\rho_i > 0$
- State and input dilations $\Lambda_{\mathbf{r}}$ and Λ_ρ.

Weighted Homogeneity for (set-valued) vector fields with inputs.

A vector field $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ (a set-valued vector field $f : \mathbb{R}^n \times \mathbb{R}^m \rightrightarrows \mathbb{R}^n$) is $(\mathbf{r}, \rho)$-homogeneous with *degree* $\delta \in \mathbb{R}$ iff $\forall \lambda > 0$:

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Weighted Homogeneity for (set-valued) vector fields with inputs.

A vector field $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ (a set-valued vector field $f : \mathbb{R}^n \times \mathbb{R}^m \rightrightarrows \mathbb{R}^n$) is $(\mathbf{r}, \rho)$-homogeneous with *degree* $\delta \in \mathbb{R}$ iff $\forall \lambda > 0$:

$$f(\Lambda_\mathbf{r} x, \Lambda_\rho u) = \lambda^\delta \Lambda_\mathbf{r} f(x, u).$$

Weighted Homogeneity with inputs

Consider a System with inputs $u \in \mathbb{R}^m$

$$\dot{x} = f(x, u)\,.$$

- State and input weight vectors $\mathbf{r} = (r_1, \cdots, r_n)$, $r_i > 0$, $\rho = (\rho_1, \cdots, \rho_m)$, $\rho_i > 0$
- State and input dilations $\Lambda_{\mathbf{r}}$ and Λ_ρ.

Weighted Homogeneity for (set-valued) vector fields with inputs.

A vector field $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ (a set-valued vector field $f : \mathbb{R}^n \times \mathbb{R}^m \rightrightarrows \mathbb{R}^n$) is $(\mathbf{r}, \rho)$-homogeneous with *degree* $\delta \in \mathbb{R}$ iff $\forall \lambda > 0$:

$$f(\Lambda_{\mathbf{r}} x, \Lambda_\rho u) = \lambda^\delta \Lambda_{\mathbf{r}} f(x, u).$$

Associated with the (set-valued) vector field $f(x, u)$ is the Differential Equation $\dot{x} = f(x, u)$ (DI $\dot{x} \in f(x, u)$), and it has a flow (solution) $\varphi(t, x, u)$.

Homogeneity of vector field ⇒ Homogeneity of Flow

If f is homogeneous then $\forall \lambda > 0$:

$$\varphi(t, \Lambda_{\mathbf{r}} x, \Lambda_{\rho} u(\lambda^{\delta} \cdot)) = \Lambda_{\mathbf{r}} \varphi(\lambda^{\delta} t, x, u(\cdot))$$

Dynamic interpretation of homogeneity

- System $\dot{x} = f(x, u)$ is homogeneous of degree δ, i.e. $f(\Lambda_{\mathbf{r}} x, \Lambda_\rho u) = \lambda^\delta \Lambda_{\mathbf{r}} f(x, u)$.
- State and input Transformation $z = \Lambda_{\mathbf{r}} x$, $w = \Lambda_\rho u$

$$\dot{z} = \Lambda_{\mathbf{r}} \dot{x} = \Lambda_{\mathbf{r}} f(x, u) = \lambda^{-\delta} f(\Lambda_{\mathbf{r}} x, \Lambda_\rho u)$$

and therefore

$$\frac{dz}{d(\lambda^\delta t)} = f(z, w)$$

- System $\dot{x} = f(x)$ is invariant under the transformation

$$G_\lambda : (t, x, u) \mapsto (\lambda^{-\delta} t, \Lambda_{\mathbf{r}} x, \Lambda_\rho u) .$$

- Internal stability $\Rightarrow$ external stability (iISS, ISS) [Bernuau et al. 2013]

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Weighted Homogeneity and Precision

Consider a System with a scalar and constant input $u \in \mathbb{R}$

$$\dot{x} = f(x, u)\,.$$

so that

$$\varphi(t, \Lambda_{\mathbf{r}} x, \lambda^{\rho} u) = \Lambda_{\mathbf{r}} \varphi(\lambda^{\delta} t, x, u)\,.$$

Suppose that asymptotically or after a finite time for some u_0

$$\lim_{t \to \infty} |\varphi_i(t, x, u_0)| = |\varphi_{\infty i}(u_0)| \leq a_i\,.$$

Therefore (using $\lambda = \left(\frac{u}{u_0}\right)^{\frac{1}{\rho}}$)

$$|\varphi_{\infty i}(u)| = |\varphi_{\infty i}(\lambda^{\rho} u_0)| = \lambda^{r_i} |\varphi_{\infty i}(u_0)| \leq \nu_i u^{\frac{r_i}{\rho}}\,,$$

with $\nu_i = a_i u_0^{-\frac{r_i}{\rho}}$.

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Homogeneous Approximation

- First Lyapunov's Theorem: $x = 0$ is LAS for $\dot{x} = f(x)$ if $x = 0$ is AS for linearized system $\dot{x} = Ax$, where $A = \frac{\partial f(0)}{\partial x}$.
- This is not true for Taylor Approximations. Example from Bacciotti & Rosier, 2005:

$$\dot{x}_1 = -x_1^3 + x_2^3\,,$$
$$\dot{x}_2 = -x_1 + x_2^5\,.$$

  $x = 0$ is GAS for approximation of order 3 (black), but it is not AS with perturbation (red) of higher order 5!
- However, it is true for homogeneous approximations (Not unique!)

$$\dot{x}_1 = -x_1^3 + x_2\,,$$
$$\dot{x}_2 = -x_1^5 + x_2^2\,.$$

  $x = 0$ is GAS for homogeneous approximation $r_1 = 1$, $r_2 = 3$, $\delta = 2$ (black), and it is still AS with perturbation (red), which is homogeneous of degree 3.

Homogeneous Domination

Typical control (similar for observation) problem:

$$\begin{aligned} \dot{x}_1 &= x_2 + f_1(x_1)\,, \\ \dot{x}_2 &= x_3 + f_2(x_1, x_2)\,, \\ &\vdots \\ \dot{x}_n &= u + f_n(x)\,. \end{aligned}$$

Homogeneous Approximation (Black):

$$\begin{aligned} \dot{x}_1 &= x_2\,, \\ &\vdots \\ \dot{x}_n &= u = \phi(x)\,. \end{aligned}$$

It is homogeneous of degree $\delta = \{-1, 0, +1\}$ and weights $r = (r_1 + \delta, r_1 + 2\delta, \cdots, r_1 + n\delta)$: NOT UNIQUE! $\Rightarrow$ Fixed by selection of the control law $\phi(x)$ $\Rightarrow$ Domination of other terms f_i.

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Alternative *Integral + state feedback* controllers for System

$$\begin{aligned}\dot{x}_1 &= x_2 \\ \dot{x}_2 &= u + \rho(t)\ ,\end{aligned}$$

- Linear Integral + state feedback controller (Homogeneous)

$$\begin{aligned}u &= -k_1 x_1 - k_2 x_2 + x_3 \\ \dot{x}_3 &= -k_3 x_1\end{aligned}$$

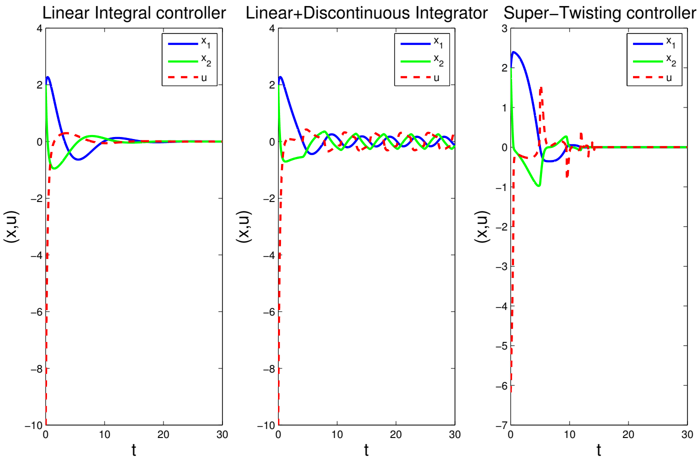
- Linear state feedback + Discontinuous Integral controller (Not Homogeneous)

$$\begin{aligned}u &= -k_1 x_1 - k_2 x_2 + x_3 \\ \dot{x}_3 &= -k_3 \mathrm{sign}(x_1)\end{aligned}$$

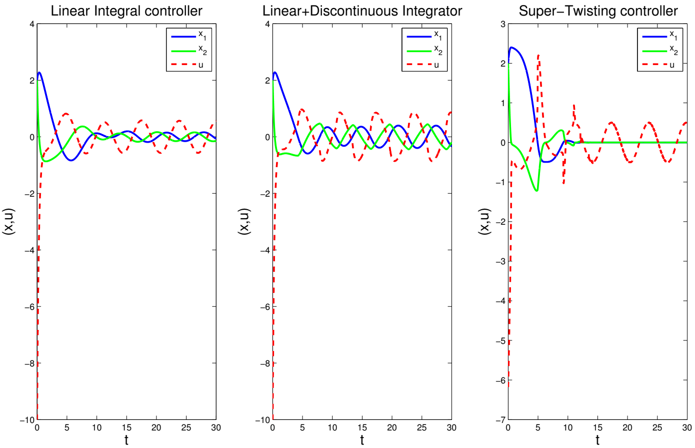
- Discontinuous I-Controller (Extended Super-Twisting) (Homogeneous)

$$\begin{aligned}u &= -k_1 |x_1|^{\frac{1}{3}} \mathrm{sign}(x_1) - k_2 |x_2|^{\frac{1}{2}} \mathrm{sign}(x_2) + x_3 \\ \dot{x}_3 &= -k_3 \mathrm{sign}(x_1)\end{aligned}$$

Controller without perturbation



Controller with perturbation



Overview

Homogeneous Design of HOSM [11]

$$\sum_{DI} : \begin{cases} \dot{x}_i = x_{i+1},\ i = 1, ..., \rho - 1, \\ \dot{x}_\rho \in [-C,\ C] + [K_m,\ K_M]u\,. \end{cases}$$

$$u = \vartheta_r(x_1,\ x_2, \cdots,\ x_\rho)\,,$$

- ϑ_r homogeneous of degree 0 (discontinuous at $x = 0$) with weights $\mathbf{r}_s = (\rho, \rho - 1, ..., 1)$

$$\vartheta_r\left(\epsilon^\rho x_1,\ \epsilon^{\rho-1}x_2,\ \ldots,\ \epsilon x_\rho\right) = \vartheta_r\left(x_1,\ x_2,\ \ldots,\ x_\rho\right)\ \ \forall\epsilon > 0$$

  - Local boundedness $\Rightarrow$ global boundedness
- System $\sum_{DI}$ is homogeneous of degree -1 with weights $\mathbf{r}_s$
  - Local contractive $\Leftrightarrow$ Global, uniformly Finite-Time stability
  - Robustness of stability $\Rightarrow$ Accuracy with respect to homogeneous perturbations $|x_i| \leq \gamma_i \tau^{\rho+1-i} = O\left(\tau^{\rho+1-i}\right)$, γ_i constants.

Concrete Homogeneous HOSM Controllers

Notation: $\lceil x \rfloor^{p} = |x|^{p} \text{sign}\,(x)$.

Nested Sliding Controllers (NSC)

- $u_{2L} = -k_2 \left\lceil x_2 + \beta_1 \lceil x_1 \rfloor^{\frac{1}{2}} \right\rfloor^{0}$,
- $u_{3L} = -k_3 \left\lceil x_3 + \beta_2 (|x_2|^3 + |x_1|^2)^{\frac{1}{6}} \left\lceil x_2 + \beta_1 \lceil x_1 \rfloor^{\frac{2}{3}} \right\rfloor^{0} \right\rfloor^{0}$,

Quasi-Continuous Sliding Controllers (QCSC)

- $u_{2C} = -k_2 \frac{\left(x_2 + \beta_1 \lceil x_1 \rfloor^{1/2}\right)}{|x_2| + \beta_1 |x_1|^{1/2}}$
- $u_{3C} = -k_3 \frac{x_3 + \beta_2 \left(|x_2| + \beta_1 |x_1|^{3/2}\right)^{-1/2} \left(x_2 + \beta_1 \lceil x_1 \rfloor^{3/2}\right)}{|x_3| + \beta_2 \left(|x_2| + \beta_1 |x_1|^{3/2}\right)^{1/2}}$.

Scaling the gains: again homogeneity!

Example: 2nd Order Nested Controller

$$\begin{aligned}\dot{x}_1 &= x_2 \\ \dot{x}_2 &= -k_2 \left\lceil x_2 + \beta_1 \lceil x_1 \rfloor^{\frac{1}{2}} \right\rfloor^0\end{aligned}$$

Linear change of coordinates: $z = \lambda^2 x$, $0 < \lambda \in \mathbb{R}$

$$\begin{aligned}\dot{z}_1 &= \lambda^2 x_2 = z_2 \\ \dot{z}_2 &= -\lambda^2 k_2 \left\lceil \frac{z_2}{\lambda^2} + \beta_1 \left\lceil \frac{z_1}{\lambda^2} \right\rfloor^{\frac{1}{2}} \right\rfloor^0 \\ &= -\lambda^2 k_2 \left\lceil \frac{z_2}{\lambda} + \beta_1 \lceil z_1 \rfloor^{\frac{1}{2}} \right\rfloor^0\end{aligned}$$

Preserves stability!

Example: 3rd Order Nested Controller

$$\begin{aligned}\dot{x}_1 &= x_2,\ \dot{x}_2 = x_3,\\ \dot{x}_3 &= -k_3\left\lceil x_3 + \beta_2(|x_2|^3 + |x_1|^2)^{\frac{1}{6}}\left\lceil x_2 + \beta_1\lceil x_1\rfloor^{\frac{2}{3}}\right\rfloor^0\right\rfloor^0\end{aligned}$$

$z = \lambda^3 x$, $0 < \lambda \in \mathbb{R} \Rightarrow$ Preserves stability!

$$\begin{aligned}\dot{z}_1 &= z_2,\ \dot{z}_2 = z_3,\\ \dot{z}_3 &= -\lambda^3 k_3\left\lceil \frac{z_3}{\lambda^3} + \beta_2\left(\left|\frac{z_2}{\lambda^3}\right|^3 + \left|\frac{z_1}{\lambda^3}\right|^2\right)^{\frac{1}{6}}\left\lceil \frac{z_2}{\lambda^3} + \beta_1\left\lceil \frac{z_1}{\lambda^3}\right\rfloor^{\frac{2}{3}}\right\rfloor^0\right\rfloor^0\\ &= -\lambda^3 k_3\left\lceil \frac{z_3}{\lambda^2} + \beta_2\left(\left|\frac{z_2}{\lambda}\right|^3 + |z_1|^2\right)^{\frac{1}{6}}\left\lceil \frac{z_2}{\lambda} + \beta_1\lceil z_1\rfloor^{\frac{2}{3}}\right\rfloor^0\right\rfloor^0\end{aligned}$$

Balance

Advantages:

- Beautiful and powerful theory: more qualitative than quantitative
- Simple: local = global, convergence = finite-time = robust
- scaling the gains (for nested controllers) for convergence acceleration

$$u = \lambda^{\rho} \vartheta_r \left(x_1, \frac{x_2}{\lambda}, \ldots, \frac{x_\rho}{\lambda^{\rho-1}} \right), \;\; \lambda > 1$$

Balance....

Limitations:

- Beyond homogeneity unclear how to design ϑ_r!
- It does not provide quantitative results, e.g.
  - Stabilizing Gains
  - Convergence Time estimation
  - Accuracy Gains
  - Performance quantities
- Behavior with respect to non homogeneous perturbations
- Behavior under interconnection
- Design for performance

Due to Limitations we require other methods, e.g. Lyapunov (but not exclusively)

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Motivation

Lyapunov based methods

- Standard methods in nonlinear control theory
- Design for robustness, optimality, performance, etc.
- Gain tuning methods
- Lyapunov Function for Internal Stability and Lyapunov-like Functions for External Stability (e.g. ISS, iISS,...)
- LF for design: Control Lyapunov Functions (CLF)
- Interconnection analysis is possible
- Robustness analysis: noise, uncertainties, perturbations

Objective:

A Lyapunov Based framework for HOSM

Belief: Combination of Homogeneity and LF $\Rightarrow$ powerful tool!

Main Problem: Construction of LF

Existence of LF for $f(x)$ Filippov Differential Inclusion

- It exists a smooth p.d. $V(x)$ [Clarke et al. 1998]
- $f(x)$ homogeneous $\Rightarrow V(x)$ homogeneous [Nakamura et al. 2002, Bernuau et al. 2014]

There are basically two issues:

- What is the form or structure of the LF?
- How to decide if $V(x)$ and $W(x)$ are positive definite (p.d.)?

There are many works on these general topics. But there are few for HOSM algorithms and taking advantage of the homogeneity properties.

State of the art

Orlov, 2005 Weak LF for Twisting Algorithm. Mechanical Energy.

Moreno and Osorio, 2008 Strong non smooth LF for Super–Twisting. No method.

Polyakov and Poznyak, 2009 Strong non smooth LF for Twisting and ST. Zubov's Method.

Santiesteban et al., 2010 Strong LF for Twisting with linear terms. No method.

Polyakov and Poznyak, 2012 Strong LF for Twisting, Terminal and Suboptimal. Zubov's Method.

Sánchez and Moreno, 2012 Strong non smooth LF for Twisting, Terminal and a sign controller. Trajectory Integration method.

Some attempts (in our group)

- First steps: the quadratic form approach
- Young's inequality and extensions
- Trajectory integration
- Reduction method
- Generalized Forms approach
- Homogenous Control Lyapunov Functions
- etc.

I will not talk about Lille's Group Implicit Lyapunov Functions (ILF) approach (Polyakov, Efimov, Perruquetti,...)!

Part II

Lyapunov-Based Design of Higher-Order Sliding Mode (HOSM) Controllers

Outline

Robust Stabilization Problem (E. Cruz)

Perturbed nonlinear system

$$\dot{x} \in F(x) + g(x)\,\xi(x)\,u\,,$$

- $x \in \mathbb{R}^n$, $u \in \mathbb{R}$
- $g(x)$ known vector field
- $F(x)$ set-vector field, ξ multivalued $\Rightarrow$ Uncertainties.

Assumptions

- $0 < K_m \le \xi(x) \le K_M$
- F, g are **r**-homogeneous of degree l.

[5, 6, 13]

Overview

Control Lyapunov Function (CLF)

(Homogeneous) CLF

$V(x) \in C^1$, p.d., $\mathbf{r}$-homogeneous of degree $m > -l$

$$\frac{\partial V(x)}{\partial x} g(x) = 0 \Rightarrow \sup_{v \in F(x)} L_v V < 0,\ \forall x \in \mathbb{R}^n \setminus \{0\}\,.$$

$\mathbf{r}$-homogeneous of degree 0 (discontinuous) Controllers

$$\begin{aligned} u &= -k\varphi_1(x) = -k\left\lceil L_{g(x)} V(x) \right\rfloor^0 , \\ u &= -k\varphi_2(x) = -k \frac{L_{g(x)} V(x)}{\|x\|_{\mathbf{r},p}^{l+m}} , \end{aligned}$$

If $k \geq k^*$, $x = 0$ is GAS, and if $l < 0$ it is Finite-Time Stable.

$$\|x\|_{\mathbf{r},p} = \left(|x_1|^{\frac{p}{r_1}} + \cdot + |x_n|^{\frac{p}{r_n}} \right)^{\frac{1}{p}},\ p \geq \max r_i,\ \text{ homogeneous norm}\,.$$

HOSM design based on CLF

The Basic Uncertain System

$$\sum_{DI} : \begin{cases} \dot{x}_i = x_{i+1},\ i = 1, ..., \rho - 1, \\ \dot{x}_\rho \in [-C, C] + [K_m, K_M]u\,. \end{cases}$$

- The Design is reduced to find a CLF.
- By a Back-Stepping-like procedure construct a CLF

Define

$$\bar{x}_i^T = [x_1, \cdots, x_i] \ ,$$

$$\mathbf{r} = (r_1, r_2, ..., r_\rho) = (\rho, \rho - 1, ..., 1) \ ,$$

$$\alpha_\rho \geq \alpha_{\rho-1} \geq \cdots \geq \alpha_1 \geq \rho\,,$$

$$m \geq r_i + \alpha_{i-1}\,.$$

Family of CLFs

CLF **r**-homogeneous of degree m: $\forall \gamma_i > 0$, $\exists k_i > 0$

$$V(x) = \gamma_\rho W_\rho(\bar{x}_\rho) + \cdots + \gamma_i W_i(\bar{x}_i) + \cdots + \gamma_1 \frac{\rho}{m} |x_1|^{\frac{m}{\rho}}$$

$$W_i(\bar{x}_i) = \frac{r_i}{m} |x_i|^{\frac{m}{r_i}} - \lceil \nu_{i-1} \rfloor^{\frac{m-r_i}{r_i}} x_i + \left(1 - \frac{r_i}{m}\right) |\nu_{i-1}|^{\frac{m}{r_i}} ,$$

$$\nu_i(\bar{x}_i) = -k_i \lceil \sigma_i \rfloor^{\frac{r_{i+1}}{\alpha_i}} , \ \sigma_1 = \lceil x_1 \rfloor^{\frac{\alpha_1}{r_1}}$$

$$\sigma_i(\bar{x}_i) = \lceil x_i \rfloor^{\frac{\alpha_i}{r_i}} - \lceil \nu_{i-1} \rfloor^{\frac{\alpha_i}{r_i}} = \lceil x_i \rfloor^{\frac{\alpha_i}{r_i}} + k_{i-1}^{\frac{\alpha_i}{r_i}} \lceil \sigma_{i-1} \rfloor^{\frac{\alpha_i}{\alpha_{i-1}}} ,$$

$V(x)$ is a continuously differentiable and **r**-homogeneous CLF of degree m.

Overview

HOSM Controllers

HOSM Discontinuous Controller

Discontinuity at $\sigma_\rho(x) = 0$

$$u_D(x) = -k_\rho \lceil \sigma_\rho(x) \rfloor^0 , k_\rho \gg 0 ,$$

HOSM Quasi-Continuous Controller

Discontinuity only at $x = 0$

$$u_Q(x) = -k_\rho \frac{\sigma_\rho(x)}{M(x)} , k_\rho \gg 0 ,$$

$M(x)$ is any continuous **r**-homogeneous positive definite function of degree α_ρ.

ρ-th order sliding mode $x = 0$ is established in Finite-Time.

Convergence Time Estimation

Convergence Time Estimation

$$T\left(x_0\right) \leq m\tau_\rho V_\rho^{\frac{1}{m}}\left(x_0\right) \ ,$$

where τ_ρ is a function of the gains $(k_1, ..., k_\rho)$, K_m and C.

Variable-Gain HOSM Controller

If $C = \bar{C} + \Theta(t, z)$, with constant $\bar{C}$ and time-varying $\Theta(t, z) \geq 0$ known.

Variable-Gain Controller

The Discontinuous Variable-Gain HOSM Controller

$$u_D(x) = -\left(K(t, z) + k_\rho\right) \lceil \sigma_\rho(x) \rfloor^0 \,, k_\rho \gg 0 \,,$$

and the Quasi-Continuous Variable-Gain HOSM Controller

$$u_Q(x) = -\left(K(t, z) + k_\rho\right) \frac{\sigma_\rho(x)}{M(x)} \,, k_\rho \gg 0 \,,$$

stabilize the origin $x = 0$ in Finite-Time if $k_\rho \gg 0$ and $K_m K(t, z) \geq \Theta(t, z)$.

Overview

Discontinuous Nested HOSM Controllers

For $\alpha_\rho \geq \cdots \geq \alpha_1 \geq \rho$, the discontinuous Controllers of orders $\rho = 2,\, 3,\, 4$ are given by

$$
\begin{aligned}
u_{2D} &= -k_2 \left[\lceil x_2 \rfloor^{\alpha_2} + k_1^{\alpha_2} \lceil x_1 \rfloor^{\frac{\alpha_2}{2}} \right]^0 \\
u_{3D} &= -k_3 \left[\lceil x_3 \rfloor^{\alpha_3} + k_2^{\alpha_3} \left[\lceil x_2 \rfloor^{\frac{\alpha_2}{2}} + k_1^{\frac{\alpha_2}{2}} \lceil x_1 \rfloor^{\frac{\alpha_2}{3}} \right]^{\frac{\alpha_3}{\alpha_2}} \right]^0 \\
u_{4D} &= -k_4 \left[\lceil x_4 \rfloor^{\alpha_4} + k_3^{\alpha_4} \left[\lceil x_3 \rfloor^{\frac{\alpha_3}{2}} + k_2^{\frac{\alpha_3}{2}} \left[\lceil x_2 \rfloor^{\frac{\alpha_2}{3}} + k_1^{\frac{\alpha_2}{3}} \lceil x_1 \rfloor^{\frac{\alpha_2}{4}} \right]^{\frac{\alpha_3}{\alpha_2}} \right]^{\frac{\alpha_4}{\alpha_3}} \right]^0
\end{aligned}
$$

and are, in general, of the type of the Nested Sliding Controllers (Levant 2005).

Discontinuous Relay Polynomial HOSM Controllers

Cruz-Zavala & Moreno (2014, 2016) [4, 3, 5]; Ding, Levant & Li (2015,2016) [7, 8].
For $\alpha_\rho = \alpha_{\rho-1} = \cdots = \alpha_1 = \alpha \geq \rho$ discontinuous "relay polynomial" controllers

$$
\begin{aligned}
u_{2R} &= -k_2 \mathrm{sign}\left(\lceil x_2 \rfloor^{\alpha} + \bar{k}_1 \lceil x_1 \rfloor^{\frac{\alpha}{2}} \right) , \\
u_{3R} &= -k_3 \mathrm{sign}\left(\lceil x_3 \rfloor^{\alpha} + \bar{k}_2 \lceil x_2 \rfloor^{\frac{\alpha}{2}} + \bar{k}_1 \lceil x_1 \rfloor^{\frac{\alpha}{3}} \right) \\
u_{4R} &= -k_4 \mathrm{sign}\left(\lceil x_4 \rfloor^{\alpha} + \bar{k}_3 \lceil x_3 \rfloor^{\frac{\alpha}{2}} + \bar{k}_2 \lceil x_2 \rfloor^{\frac{\alpha}{3}} + \bar{k}_1 \lceil x_1 \rfloor^{\frac{\alpha}{4}} \right)
\end{aligned}
$$

where for $\rho = 2$, $\bar{k}_1 = k_1^{\alpha}$; for $\rho = 3$, $\bar{k}_1 = k_2^{\alpha} k_1^{\frac{\alpha}{2}}$, $\bar{k}_2 = k_2^{\alpha}$; and for general ρ, $\bar{k}_i = \prod_{j=i}^{\rho-1} k_j^{\frac{\alpha}{\rho-j}}$.

Quasi-Continuous Nested or Relay Polynomial HOSM Controllers

For arbitrary $\beta_i > 0$

$$
\begin{aligned}
u_{2Q} &= -k_2 \frac{\lceil x_2 \rfloor^{\alpha_2} + k_1^{\alpha_2} \lceil x_1 \rfloor^{\frac{\alpha_2}{2}}}{|x_2|^{\alpha_2} + \beta_1 |x_1|^{\frac{\alpha_2}{2}}}\,, \\
u_{3Q} &= -k_3 \frac{\lceil x_3 \rfloor^{\alpha_3} + k_2^{\alpha_3} \left\lceil \lceil x_2 \rfloor^{\frac{\alpha_2}{2}} + k_1^{\frac{\alpha_2}{2}} \lceil x_1 \rfloor^{\frac{\alpha_2}{3}} \right\rfloor^{\frac{\alpha_3}{\alpha_2}}}{|x_3|^{\alpha_3} + \beta_2 |x_2|^{\frac{\alpha_3}{2}} + \beta_1 |x_1|^{\frac{\alpha_3}{3}}} \\
u_{4Q} &= -k_4 \frac{\lceil x_4 \rfloor^{\alpha_4} + k_3^{\alpha_4} \left\lceil \lceil x_3 \rfloor^{\frac{\alpha_3}{2}} + k_2^{\frac{\alpha_3}{2}} \left\lceil \lceil x_2 \rfloor^{\frac{\alpha_2}{3}} + k_1^{\frac{\alpha_2}{3}} \lceil x_1 \rfloor^{\frac{\alpha_2}{4}} \right\rfloor^{\frac{\alpha_3}{\alpha_2}} \right\rfloor^{\frac{\alpha_4}{\alpha_3}}}{|x_4|^{\alpha_4} + \beta_3 |x_3|^{\frac{\alpha_4}{2}} + \beta_2 |x_2|^{\frac{\alpha_4}{3}} + \beta_1 |x_1|^{\frac{\alpha_4}{4}}}
\end{aligned}
$$

Overview

Numerical Gain Calculation

- k_i, for $i = 1, \cdots, \rho - 1$, selected to render $V(x)$ a CLF,
- k_ρ to obtain $\dot{V} < 0$.
- Fix m, ρ, α_i and γ_i. Set $k_1 > 0$, for $i = 2, \cdots, \rho$

$$k_i > \max_{\bar{x}_i \in S_i} \{\Phi_i(\bar{x}_i)\} =: G_i(k_1, \cdots, k_{i-1}), \tag{2}$$

- Maximization feasible since
  1. Φ is $\mathbf{r}$-homogeneous of degree 0:achieves all its values on the homogeneous unit sphere $S_i = \{\bar{x}_i \in \mathbb{R}^i : \|\bar{x}_i\|_{\mathbf{r},p} = 1\}$, and
  2. Φ is upper-semicontinuous $\Rightarrow$ it has a maximum on S_i.
- Parametrization

$$k_i = \mu_i k_1^{\frac{\rho}{\rho-(i-1)}}, \; k_\rho > \frac{1}{K_m}\left(\mu_\rho k_1^\rho + C\right), \tag{3}$$

for some positive constants μ_i independent of k_1.
- Parametrization can be used for all controllers, but k_ρ different for discontinuous and quasi-continuous controllers.

Analytical Gain calculation

It is possible (but cumbersome) to provide for any order an analytical estimation of the values of the gains using classical inequalities.

The simplest case with $\rho = 3$, u_{3D}, u_{3R} and u_{3Q}

For any values of $\alpha_3 \geq \alpha_2 \geq r_1 = 3$, $m \geq r_2 + \alpha_2$, $\gamma_1 > 0$, $0 < \eta < 1$ and $k_1 > 0$,

$$k_2 > \frac{r_2 2^{\frac{m-2r_2}{\alpha_2}}}{m-1} \left(\frac{(m-r_1) 2^{\frac{\alpha_2 - r_2}{\alpha_2}}}{m-1} \right)^{\frac{m-r_1}{r_2}} \frac{\left(\gamma_1 + \frac{m-r_2}{r_1} k_1^{\frac{m}{r_2}} \right)^{\frac{m-1}{r_2}}}{(\eta \gamma_1 k_1)^{\frac{m-r_1}{r_2}}} .$$

Remarks

- By homogeneity the gain scaling with any $L \geq 1$

  $$\mathbf{k}^T = (k_1, \cdots, k_\rho) \to \mathbf{k}_L^T = (L^{\frac{1}{\rho}} k_1, \cdots, L^{\frac{1}{\rho+1-i}} k_i, \cdots, L k_\rho)$$

  preserves the stability for any α_j.
- Convergence will be accelerated for $L > 1$, or the size of the allowable perturbation C will be incremented to LC.
- The gains obtained by means of the LF can be very large for practical applications, so that a simulation-based gain design is eventually necessary (see [5]).
- The gain design problem is an important and unexplored one.

Overview

Example

Kinematic model of a car [10]

$$\dot{z}_1 = v\cos(z_3)\,,\ \ \dot{z}_2 = v\sin(z_3)\,,\ \ \dot{z}_3 = \left(\tfrac{v}{L}\right)\tan(z_4)\,,\ \ \dot{z}_4 = u,$$

- z_1, z_2: cartesian coordinates of the rear-axle middle point,
- z_3: the orientation angle,
- z_4: the steering angle, (Actual control)
- v: the longitudinal velocity ($v = 10$ m.s^{-1}),
- L: distance between the two axles ($L = 5$ m), and
- u: the control input.

u is used as a new control input in order to avoid discontinuities on z_4.

Control Objective

- Control Task: steer the car from a given initial position to the trajectory $z_{2\text{ref}} = 10\sin(0.05z_1) + 5$ in finite time.
- Turn on the controllers after Observer converged (0.5 sec.).
- Sliding variable $\sigma = z_2 - z_{2\text{ref}}$, relative degree $\rho = 3$, Model

$$\dot{x}_1 = x_2,\ \dot{x}_2 = x_3,\ \dot{x}_3 = \phi(\cdot) + \gamma(\cdot)u,$$

  where $x = \begin{bmatrix}\sigma & \dot{\sigma} & \ddot{\sigma}\end{bmatrix}^T$.
- Simulations: Euler's method, sampling time $\tau = 0.0005$.
- Bounds: $|\phi| \leq C_0 = 49.63$, $K_m = 6.38 \leq \gamma \leq K_M = 46.77$.
- Controllers: (i) Levant's Discontinuous Controller **(L3)** with $\beta_1 = 1$, $\beta_2 = 2$ and $k_3 = 20$; (ii) Levant's Quasi-Continuous Controller **(Q3)** with $\beta_1 = 1$, $\beta_2 = 2$ and $k_3 = 24.5$; and (iii) Proposed Discontinuous Controller **(E3)** $u_{3D} = -k_3\lceil\sigma_3\rfloor^0$ with $k_1 = 1$, $k_2 = 1.5$, $k_3 = 20$.

Simulations

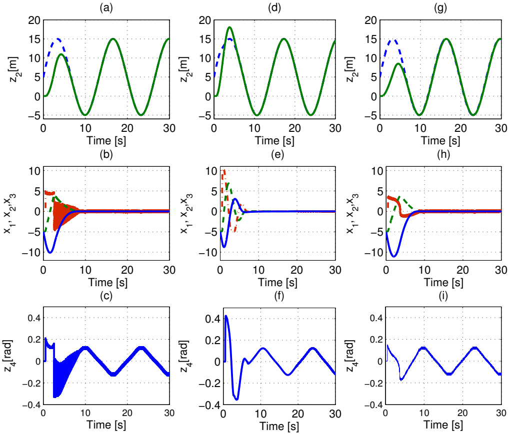


Figure : Left column: Levant's Discontinuous Controller (L3), Middle Column: Levant's Quasi-Continuous Controller (Q3); Right Column: Proposed Discontinuous Controller (E3).

Simulations

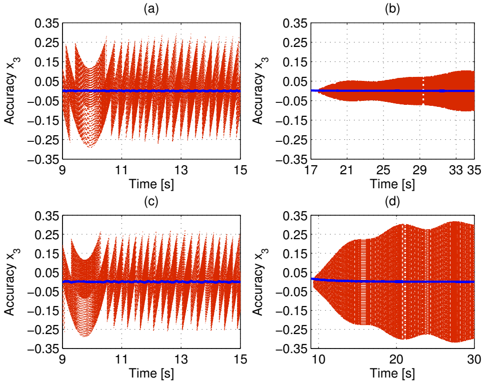


Figure : Accuracy: (a) with **(L3)**; (b) with **(Q3)**; (c) with **(E3)**; (d) with **(Q3)** ($k_3 = 70$).

Advantage of **(E3)**: combines fast convergence rate of **(L3)** with smooth transient response of **(Q3)**.

Resumé

- New: Methodological approach to design HOSM controllers using CLF.
- Different alternatives to find CLFs: Back-stepping, Polynomial methods, ...
- It can be extended to design controllers with Fixed-Time convergence.
- Drawback: Calculation of gains k_i needs maximization of 0-degree homogeneous functions.

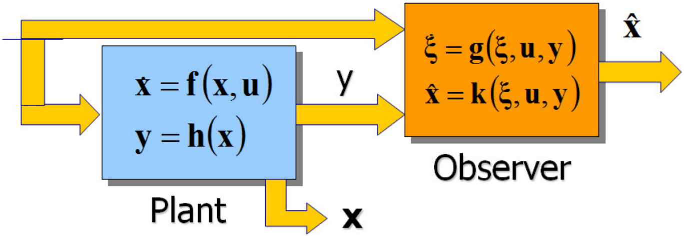
Part III

HOSM Differentiation/Observation: A Lyapunov Approach

Outline

Overview

Basic Observation Problem



$$\forall \mathbf{x}(t), \quad \lim_{t\to\infty}\left(\hat{\mathbf{x}}(t) - \mathbf{x}(t)\right) = \mathbf{0}$$

Variations of the observation problem: with unknown inputs, practical observers, robust observers, stochastic framework to deal with noises, ...

An important Property: Observability

Consider a nonlinear system without inputs, $x \in \mathbb{R}^n$, $y \in \mathbb{R}$

$$\begin{aligned} \dot{x}(t) &= f(x(t)) \ , \ x(t_0) = x_0 \\ y(t) &= h(x(t)) \end{aligned}$$

Differentiating the output

$$\begin{aligned} y(t) &= h(x(t)) \\ \dot{y}(t) &= \frac{d}{dt} h(x(t)) = \frac{\partial h(x)}{\partial x}\dot{x}(t) = \frac{\partial h(x)}{\partial x} f(x) := L_f h(x) \\ \ddot{y}(t) &= \frac{\partial L_f h(x)}{\partial x}\dot{x}(t) = \frac{\partial L_f h(x)}{\partial x} f(x) := L_f^2 h(x) \\ &\vdots \\ y^{(n-1)}(t) &= \frac{\partial L_f^{n-2} h(x)}{\partial x}\dot{x}(t) = \frac{\partial L_f^{n-2} h(x)}{\partial x} f(x) := L_f^{n-1} h(x) \end{aligned}$$

where $L_f^k h(x)$ are *Lie's derivatives of* h *along* f.

Evaluating at $t = 0$

$$\begin{bmatrix} y(0) \\ \dot{y}(0) \\ \ddot{y}(0) \\ \vdots \\ y^{(k)}(0) \end{bmatrix} = \begin{bmatrix} h(x_0) \\ L_f h(x_0) \\ L_f^2 h(x_0) \\ \vdots \\ L_f^k h(x_0) \end{bmatrix} := \mathcal{O}_n(x_0)$$

$\mathcal{O}_n(x)$: Observability map

Theorem

If $\mathcal{O}_n(x)$ is injective (invertible) $\rightarrow$ The NL system is observable.

Observability Form

In the coordinates of the output and its derivatives

$$z = \mathcal{O}_n\left(x\right) \ , \ x = \mathcal{O}_n^{-1}\left(z\right)$$

the system takes the (observability) form

$$\begin{aligned}
\dot{z}_1 &= z_2 \\
\dot{z}_2 &= z_3 \\
&\vdots \\
\dot{z}_n &= \phi\left(z_1, \, z_2, \, \ldots, \, z_n\right) \\
y &= z_1
\end{aligned}$$

So we can consider a system in this form as a basic structure.

A Simple Observer and its Properties

Plant: $\dot{x}_1 = x_2 \, , \, \dot{x}_2 = w(t)$

Observer: $\dot{\hat{x}}_1 = -l_1 \left(\hat{x}_1 - x_1 \right) + \hat{x}_2 \, , \, \dot{\hat{x}}_2 = -l_2 \left(\hat{x}_1 - x_1 \right)$

Estimation Error: $e_1 = \hat{x}_1 - x_1, \, e_2 = \hat{x}_2 - x_2$

$$\dot{e}_1 = -l_1 e_1 + e_2 \, , \, \dot{e}_2 = -l_2 e_1 - w\left(t\right)$$

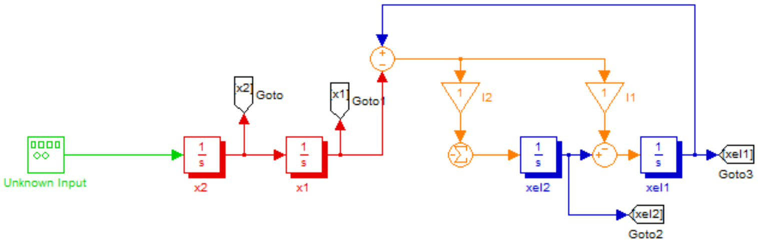


Figure : Linear Plant with an unknown input and a Linear Observer.

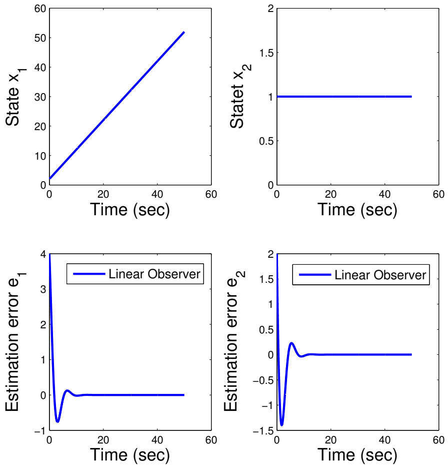
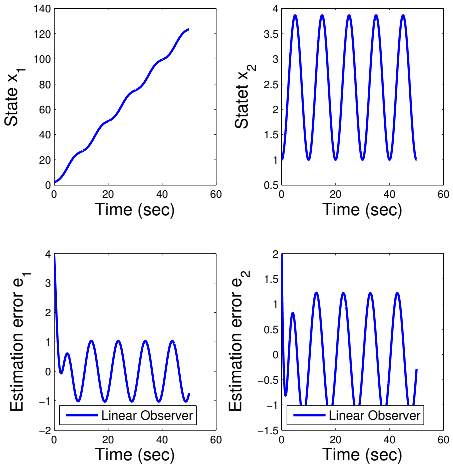


Figure : Behavior of Plant and the Linear Observer without unknown input.



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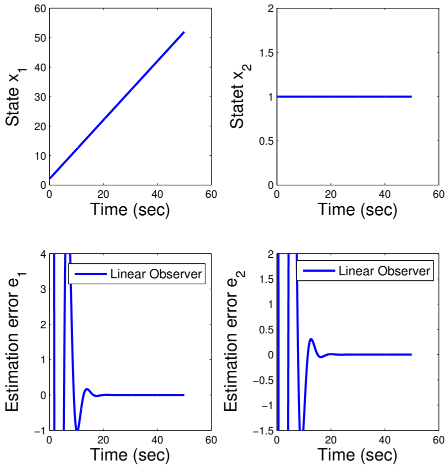

Figure : Behavior of Plant and the Linear Observer without UI with large initial conditions.

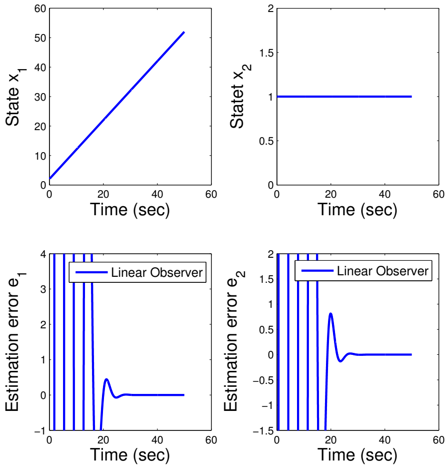

Figure : Behavior of Plant and the Linear Observer without UI with very large initial conditions.

Recapitulation.

Linear Observer for Linear Plant

- If no unknown inputs/Uncertainties: it converges exponentially fast.
- If there are unknown inputs/Uncertainties: no convergence. At best bounded error.
- Convergence time depends on the initial conditions of the observer
- Is it possible to alleviate these drawbacks?

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Sliding Mode Observer (SMO)

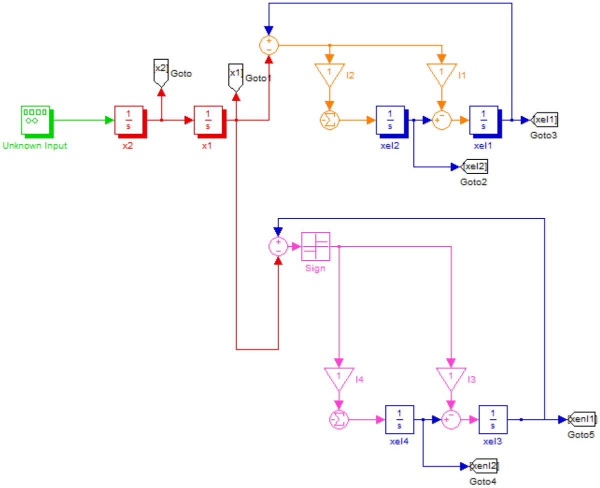

Figure : Linear Plant with an unknown input and a SM Observer.

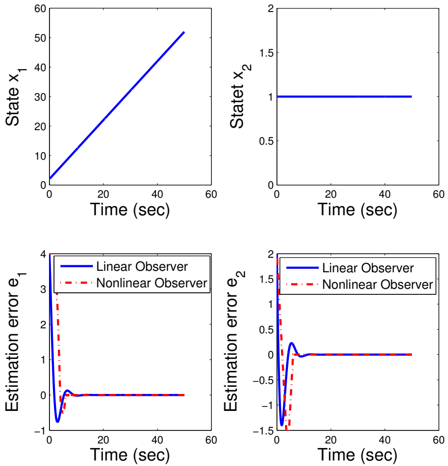

Figure : Behavior of Plant and the SM Observer without unknown input.

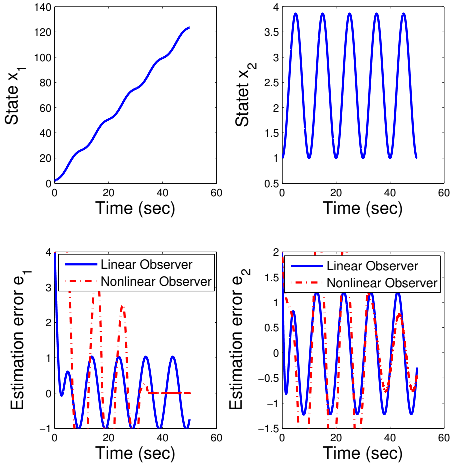

Figure : Behavior of Plant and the SM Observer with unknown input.

Recapitulation.

Sliding Mode Observer for Linear Plant

- If no unknown inputs/Uncertainties: e_1 converges in finite time, and e_2 converges exponentially fast.
- If there are unknown inputs/Uncertainties: no convergence. At best bounded error. Only e_1 converges in finite time!
- Convergence time depends on the initial conditions of the observer
- It is not the solution we expected! None of the objectives has been achieved!

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Overview

Super-Twisting Algorithm (STA)

Plant:

$$\dot{x}_1 = x_2\,,$$
$$\dot{x}_2 = w(t)$$

Observer:

$$\dot{\hat{x}}_1 = -l_1 \left|e_1\right|^{\frac{1}{2}} \operatorname{sign}\left(e_1\right) + \hat{x}_2\,,$$
$$\dot{\hat{x}}_2 = -l_2 \operatorname{sign}\left(e_1\right)$$

Estimation Error: $e_1 = \hat{x}_1 - x_1$, $e_2 = \hat{x}_2 - x_2$

$$\dot{e}_1 = -l_1 \left|e_1\right|^{\frac{1}{2}} \operatorname{sign}\left(e_1\right) + e_2$$
$$\dot{e}_2 = -l_2 \operatorname{sign}\left(e_1\right) - w\left(t\right)\,,$$

Solutions in the sense of Filippov.

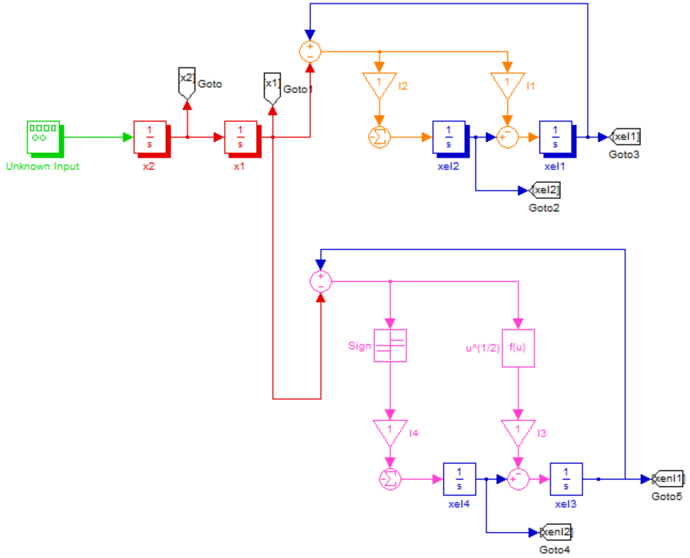

Figure : Linear Plant with an unknown input and a SOSM Observer.

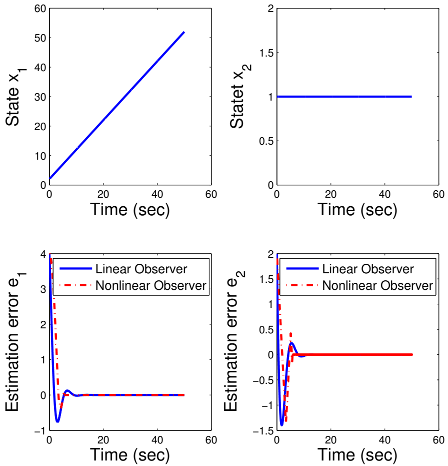

Figure : Behavior of Plant and the Non Linear Observer without unknown input.

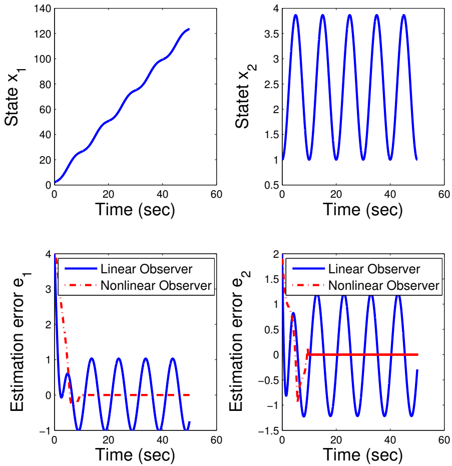

Figure : Behavior of Plant and the Non Linear Observer with unknown input.

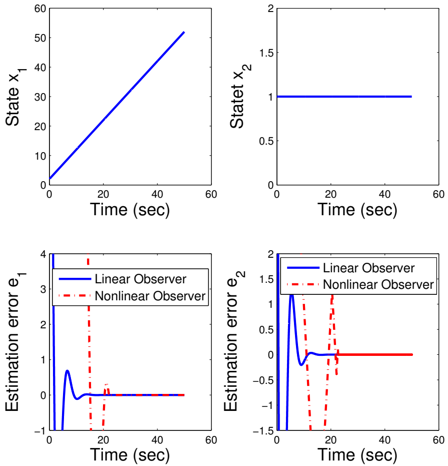

Figure : Behavior of Plant and the Non Linear Observer without UI with large initial conditions.

Recapitulation.

Super-Twisting Observer for Linear Plant

- If no unknown inputs/Uncertainties: e_1 and e_2 converge in finite-time!
- If there are unknown inputs/Uncertainties: e_1 and e_2 converge in finite-time! Observer is insensitive to perturbation/uncertainty!
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Overview

Generalized Super-Twisting Algorithm (GSTA)

Plant:

$$\dot{x}_1 = x_2\,,$$
$$\dot{x}_2 = w(t)$$

Observer:

$$\dot{\hat{x}}_1 = -l_1\phi_1\left(e_1\right) + \hat{x}_2\,,$$
$$\dot{\hat{x}}_2 = -l_2\phi_2\left(e_1\right)$$

Estimation Error: $e_1 = \hat{x}_1 - x_1$, $e_2 = \hat{x}_2 - x_2$

$$\dot{e}_1 = -l_1\phi_1\left(e_1\right) + e_2$$
$$\dot{e}_2 = -l_2\phi_2\left(e_1\right) - w\left(t\right)\,,$$

Solutions in the sense of Filippov.

$$\phi_1\left(e_1\right) = \mu_1\left|e_1\right|^{\frac{1}{2}}\operatorname{sign}\left(e_1\right) + \mu_2\left|e_1\right|^{\frac{3}{2}}\operatorname{sign}\left(e_1\right)\,,\ \mu_1\,,\ \mu_2 \geq 0\,,$$

$$\phi_2\left(e_1\right) = \frac{\mu_1^2}{2}\operatorname{sign}\left(e_1\right) + 2\mu_1\mu_2 e_1 + \frac{3}{2}\mu_2^2\left|e_1\right|^2\operatorname{sign}\left(e_1\right)\,,$$

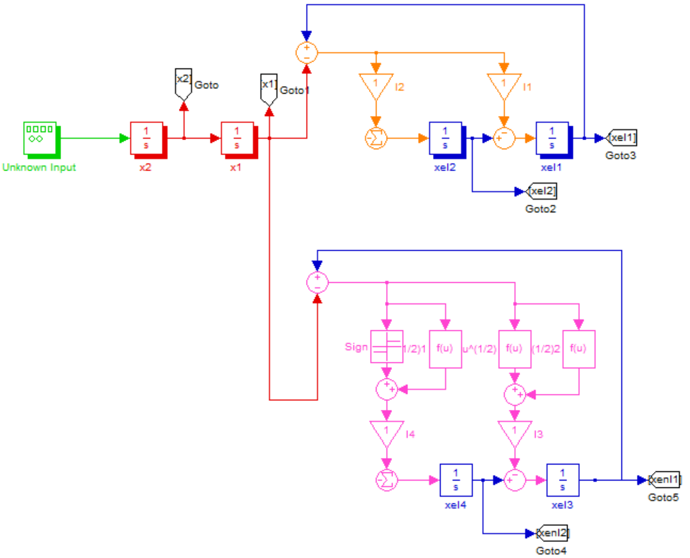

Figure : Linear Plant with an unknown input and a Non Linear Observer.

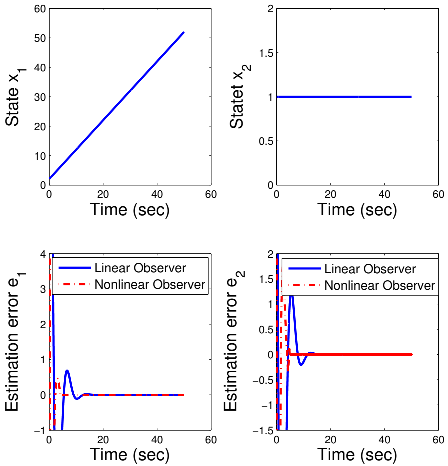

Figure : Behavior of Plant and the Non Linear Observer without unknown input and large initial conditions.

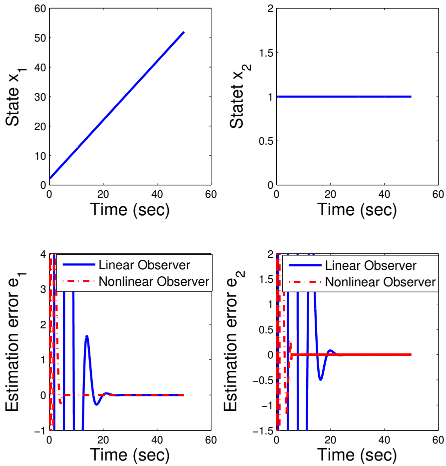

Figure : Behavior of Plant and the Non Linear Observer without unknown input and very large initial conditions.

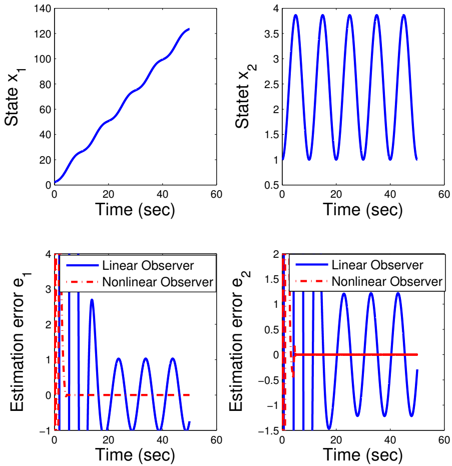

Figure : Behavior of Plant and the Non Linear Observer with UI with large initial conditions.

Effect: Convergence time independent of I.C.

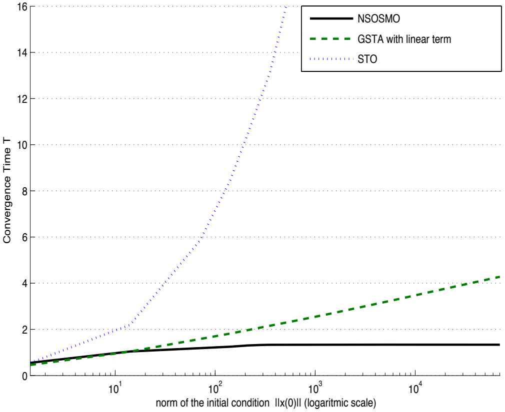

Figure : Convergence time when the initial condition grows.

Recapitulation.

Generalized Super-Twisting Observer for Linear Plant

- If no unknown inputs/Uncertainties: e_1 and e_2 converge in finite-time!
- If there are unknown inputs/Uncertainties: e_1 and e_2 converge in finite-time! Observer is insensitive to perturbation/uncertainty!
- Convergence time is independent of the initial conditions of the observer!.
- All objectives were achieved!
- How to proof these properties?

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- How to proof these properties?

What have we achieved?

An algorithm

- Robust: it converges despite of unknown inputs/uncertainties
- Exact: it converges in finite-time
- The convergence time can be preassigned for any arbitrary initial condition.
- But there is no free lunch!

It is useful for

- Observation
- Estimation of perturbations/uncertainties
- Control: Nonlinear PI-Control
- in practice?

Some Generalizations are available but Still a lot is missing

Overview

Lyapunov functions:

1. We propose a *Family* of strong Lyapunov functions, that are Quadratic-like
2. This family allows the estimation of convergence time,
3. It allows to study the robustness of the algorithm to different kinds of perturbations,
4. All results are obtained in a Linear-Like framework, known from classical control,
5. The analysis can be obtained in the same manner for a linear algorithm, the classical ST algorithm and a combination of both algorithms (GSTA), that is non homogeneous.

Generalized STA

$$
\begin{aligned}
\dot{x}_1 &= -k_1\phi_1(x_1) + x_2 \\
\dot{x}_2 &= -k_2\phi_2(x_1) \ ,
\end{aligned} \tag{4}
$$

Solutions in the sense of Filippov.

$$
\begin{aligned}
\phi_1(e_1) &= \mu_1 |e_1|^{\frac{1}{2}} \operatorname{sign}(e_1) + \mu_2 |e_1|^{q} \operatorname{sign}(e_1) \ , \ \mu_1 \ , \ \mu_2 \geq 0 \ , \ q \geq 1 \ , \\
\phi_2(e_1) &= \frac{\mu_1^2}{2} \operatorname{sign}(e_1) + \left(q + \frac{1}{2}\right) \mu_1\mu_2 |e_1|^{q-\frac{1}{2}} \operatorname{sign}(e_1) + \\
&\quad + q\mu_2^2 |e_1|^{2q-1} \operatorname{sign}(e_1) \ ,
\end{aligned}
$$

- Standard STA: $\mu_1 = 1$, $\mu_2 = 0$
- Linear Algorithm: $\mu_1 = 0$, $\mu_2 > 0$, $q = 1$.
- GSTA: $\mu_1 = 1$, $\mu_2 > 0$, $q > 1$.

Outline

Quadratic-like Lyapunov Functions

System can be written as:

$$\dot{\zeta} = \phi_1'(x_1)\, A\zeta \ , \ \zeta = \begin{bmatrix} \phi_1(x_1) \\ x_2 \end{bmatrix} \ , \ A = \begin{bmatrix} -k_1 & 1 \\ -k_2 & 0 \end{bmatrix} \ .$$

Family of *strong* Lyapunov Functions:

$$V(x) = \zeta^T P \zeta \ , \ P = P^T > 0 \ .$$

Time derivative of Lyapunov Function:

$$\dot{V}(x) = \phi_1'(x_1)\, \zeta^T \left(A^T P + PA \right) \zeta = -\phi_1'(x_1)\, \zeta^T Q \zeta$$

Algebraic Lyapunov Equation (ALE):

$$A^T P + PA = -Q$$

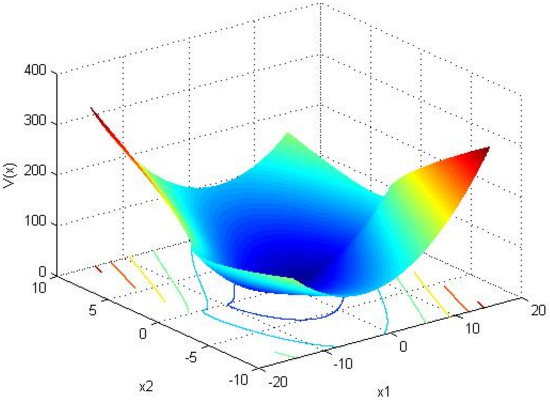

Figure : The Lyapunov function.

Lyapunov Function

Proposition

- *If A Hurwitz then $x = 0$ Finite-Time stable (if $\mu_1 = 1$) and for every $Q = Q^T > 0$, $V(x) = \zeta^T P \zeta$ is a global, strong Lyapunov function, with $P = P^T > 0$ solution of the ALE, and*

  $$\dot{V} \leq -\gamma_1 (Q, \mu_1) V^{\frac{1}{2}} (x) - \gamma_2 (Q, \mu_2) V (x) \ ,$$

  where

  $$\gamma_1 (Q, \mu_1) \triangleq \mu_1 \frac{\lambda_{\min} \{Q\} \lambda_{\min}^{\frac{1}{2}} \{P\}}{2\lambda_{\max} \{P\}} \ , \ \gamma_2 (Q, \mu_2) \triangleq \mu_2 \frac{\lambda_{\min} \{Q\}}{\lambda_{\max} \{P\}}$$

- *If A is not Hurwitz then $x = 0$ unstable.*

Convergence Time

Proposition

If $k_1 > 0$ *,* $k_2 > 0$*, and* $\mu_2 \geq 0$ *a trajectory of the GSTA starting at* $x_0 \in \mathbb{R}^2$ *converges to the origin in finite time if* $\mu_1 = 1$*, and it reaches that point at most after a time*

$$T = \begin{cases} \frac{2}{\gamma_1(Q,\mu_1)} V^{\frac{1}{2}}(x_0) & \text{if} \quad \mu_2 = 0 \\ \frac{2}{\gamma_2(Q,\mu_2)} \ln\left(\frac{\gamma_2(Q,\mu_2)}{\gamma_1(Q,\mu_1)} V^{\frac{1}{2}}(x_0) + 1\right) & \text{if} \quad \mu_2 > 0 \end{cases},$$

When $\mu_1 = 0$ *the convergence is exponential.*

For Design: T depends on the gains!

Outline

GSTA with perturbations: ARI

GSTA with time-varying and/or nonlinear perturbations

$$\begin{aligned} \dot{x}_1 &= -k_1\phi_1(x_1) + x_2 \\ \dot{x}_2 &= -k_2\phi_2(x_1) + \rho(t,x) \end{aligned} \ .$$

Assume

$$2\left|\rho(t,x)\right| \leq \delta$$

Analysis: The construction of Robust Lyapunov Functions can be done with the classical method of solving an Algebraic Ricatti Inequality (ARI), or equivalently, solving the LMI

$$\begin{bmatrix} A^T P + PA + \epsilon P + \delta^2 C^T C & PB \\ B^T P & -1 \end{bmatrix} \leq 0 \ ,$$

where

$$A = \begin{bmatrix} -k_1 & 1 \\ -k_2 & 0 \end{bmatrix} \ , \ C = \begin{bmatrix} 1 & 0 \end{bmatrix} \ , \ B = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \ .$$

Overview

Outline

Differentiation

- Signal $f(t)$ is a Lebesgue-measurable function on $[0, \infty)$.
- $f(t) = f_0(t) + v(t)$: unknown
    - $f_0(t)$, unknown base signal, n-times differentiable,
    - $|f_0|^{(n)}(t) \leq L$, L known
    - $|v(t)| \leq \eta$ uniformly bounded noise signal.
- Using: $\varsigma_1 = f_0(t), \ldots, \varsigma_{i+1} = f_0^{(i)}(t) \triangleq \frac{d^i}{dt^i} f_0(t)$, $i = 1, \ldots, n$,
- state representation of the base signal

$$
\begin{aligned}
\dot{\varsigma}_i &= \varsigma_{i+1}, \ i = 1, \cdots, n-1, \\
\dot{\varsigma}_n &= f_0^{(n)}(t) \\
y &= \varsigma_1 + v
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- Differentiator = Observer with (bounded) Unknown Input

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- Differentiator = Observer with (bounded) Unknown Input

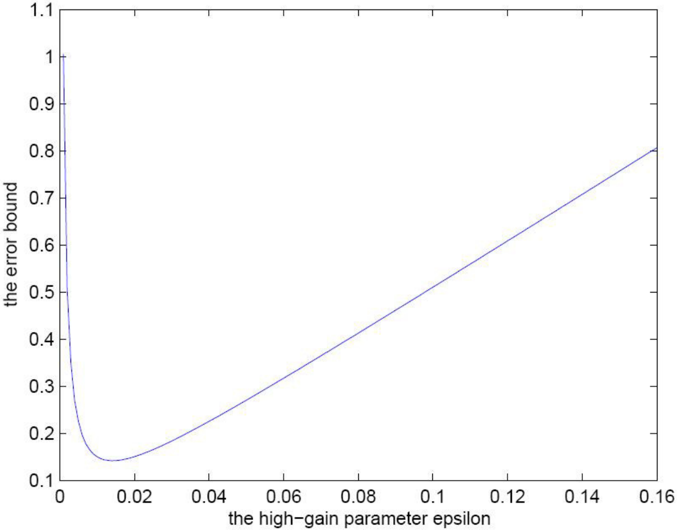
Outline

The Linear or High Gain Differentiator

$$
\begin{aligned}
\dot{x}_i &= -k_i \frac{1}{\epsilon^i}(x_1 - f) + x_{i+1}\,, \\
\vdots & \qquad i = 1, \cdots, n-1 \\
\dot{x}_n &= -k_n \frac{1}{\epsilon^n}(x_1 - f)\,,
\end{aligned}
$$

- Smooth differentiator
- Detailed analysis possible using linear methods: Vasiljevic and Khalil (2008)
- Quadratic Lyapunov Function
- Gain optimization

Trade off → Optimization



Levant's Robust and Exact Differentiator

$$
\begin{aligned}
\dot{x}_i &= -k_i \lambda^{\frac{i}{n}} \lceil x_1 - f \rfloor^{\frac{n-i}{n}} + x_{i+1}\,, \\
&\vdots \qquad i = 1, \cdots, n-1 \\
\dot{x}_n &= -k_n \lambda \lceil x_1 - f \rfloor^{0}\,,
\end{aligned}
$$

$$
\lfloor z \rceil^{p} = |z|^{p} \mathrm{sign}(z)
$$

- Levant 1998 (2nd order), 2003 (arbitrary order)
- Discontinuous: Filippov's Differential Inclusion
- In the absence of noise it converges exactly in finite time.
- Basic for Higher Order Sliding Modes. Extensions: J.P. Barbot, Fridman, ...
- Convergence proof: Geometry and Homogeneity
- It provides Qualitative properties. No gain design method.

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Outline

Homogeneous Differentiators

$$\begin{aligned} \dot{x}_i &= -k_i \lambda^{\frac{i}{n}} \lceil x_1 - f \rfloor^{\frac{r_{i+1}}{r_1}} + x_{i+1} \,, \\ &\vdots \qquad i = 1, \cdots, n-1 \\ \dot{x}_n &= -k_n \lambda \lceil x_1 - f \rfloor^{\frac{r_{n+1}}{r_1}} \,, \end{aligned}$$

$0 < r_{i+1} = r_i + d \,,\, i = 1, \ldots, n \,,\, r_n = 1,\, -1 \leq d \leq 0$

- For $d = 0$: Linear (HG) Differentiator (Khalil and Coauthors).
- For $d = -1$: Levant's Differentiator. (1998, 2003,...).
- For $-1 < d \leq 0$ Differentiator is *continuous*
- For $-1 = d$ Differentiator is *discontinuous*, i.e. (Differential Inclusion).

Homogeneous Differentiators

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\dot{x}_i &= -k_i \lambda^{\frac{i}{n}} \lceil x_1 - f \rfloor^{\frac{r_{i+1}}{r_1}} + x_{i+1} \,, \\
\vdots & \qquad i = 1, \cdots, n-1 \\
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\end{aligned}$$

$0 < r_{i+1} = r_i + d\,,\ i = 1, \ldots, n\,,\ r_n = 1,\ -1 \leq d \leq 0$

- For $d = 0$: Linear (HG) Differentiator (Khalil and Coauthors).
- For $d = -1$: Levant's Differentiator. (1998, 2003,...).
- For $-1 < d \leq 0$ Differentiator is *continuous*
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Differentiation error $e_i \triangleq x_i - f_0^{(i-1)}$

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- If $f^{(n)}(t) \equiv 0$ and $-1 < d \leq 0$ homogeneous with homogeneity degree d and weights $\mathbf{r} = [r_1, \cdots, r_n]$.
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Properties of homogeneous differentiators

Properties

Set $\lambda = 1$. $\exists k_i(L, d)$ such that as $t \to \infty$ the differentiation error e_i

- For $v(t) \equiv 0$ and polynomial signals, $\mathcal{S}_p^n = \left\{ f^{(n)}(t) \equiv 0 \right\}$
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Remarks

- For $d = 0$ this is well known. Lyapunov and Frequency analysis is possible.
- For $-\epsilon < d < 0$ Perruquetti et al. (2008) show convergence for Hurwitz gains.
- For continuous cases $-1 < d \leq 0$ there exist smooth Lyapunov functions: Yang and Lin (2004), Qian and Lin (2005), Andrieu et al. (2006,2008,2009),...
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The Lyapunov Function I

Fix $p \geq r_1 + r_2 = 2 - (2n-3)\,d > 1$ and define

$$z_i = \frac{e_i}{k_{i-1}},\ \tilde{k}_i = \frac{k_i}{k_{i-1}},\ k_0 = 1\,,\ \bar{\delta}(t) = -\frac{f^{(n)}(t)}{k_{n-1}}\,.$$

$$Z_i(z_i,\ z_{i+1}) \quad = \quad \frac{r_i}{p}\left|z_i\right|^{\frac{p}{r_i}} - z_i\left\lceil z_{i+1}\right\rfloor^{\frac{p-r_i}{r_{i+1}}} + \left(\frac{p-r_i}{p}\right)\left|z_{i+1}\right|^{\frac{p}{r_{i+1}}}\ ,$$

Z_i are continuously differentiable, positive semidefinite and $Z_i(z_i,\ z_{i+1}) = 0$ if and only if $\left\lceil z_i\right\rfloor^{\frac{p}{r_i}} = \left\lceil z_{i+1}\right\rfloor^{\frac{p}{r_{i+1}}}$.

The Lyapunov Function II

Lyapunov Function

For every $p \geq 2 - (2n-3)\, d > 1$ and $\beta_i > 0$ each differentiator of the family $-1 \leq d \leq 0$ admits a strong, proper, smooth and $\mathbf{r}-$homogeneous of degree p Lyapunov function of the form

$$\begin{aligned} V(z) &= \sum_{j=1}^{n-1} \beta_j Z_j\left(z_j,\, z_{j+1}\right) + \beta_n \frac{1}{p} \left|z_n\right|^p \\ \beta_i &> 0\,, i = 1, \cdots, n\,. \quad \square \end{aligned}$$

- $V(z)$ is positive definite and (due to homogeneity) radially unbounded.
- For the linear case ($d = 0$, $p = 2$) V is a quadratic form.

Idea of the Proof

- The basic idea (similar to Yang and Lin 2004, Qian and Lin 2005, Andrieu et al. 2006, 2008, ...) is to reduce stepwise the observer and showing convergence for a smaller observer.
- For the discontinuous case an issue: $\dot{V}$ is discontinuous. Properties of continuous homogeneous functions are not valid. Two ways out
- The derivative satisfies

$$\dot{V} \leq -\alpha_2 V^{\frac{p+d}{p}} + \alpha_3 L V^{\frac{p-1}{p}} + \sum_{i=1}^{n} \tilde{k}_i \mu_i V^{\frac{p-r_i}{p}} \left|\eta\right|^{\frac{r_{i+1}}{r_1}} ,$$

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- Using standard arguments: differentiation error is ISS with respect to the noise $\eta(t)$ and $f_0^n(t)$.

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Some observations

- Noise influence:
  - For $d = -1$: $\exists \mu_i(k_i)$, s.t.

$$|e_i| = |x_i - f_0^{i-1}| \leq \mu_i |\eta|^{\frac{n-i+1}{n}}$$

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- Gain selection: From down upwards $(k_n, k_{n-1}, \cdots, k_1)$ independent of the order. Calculation by maximization of a homogeneous function.
- Parameter $\lambda > 1$ accelerates convergence and increases the allowed bound L, but it also increases the noise effect.
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$$\dot{V} \leq -\kappa V(z)^{\frac{p-1}{p}}, \kappa > 0, , T(z_0) \leq \frac{p}{\kappa} V^{\frac{1}{p}}(z_0).$$

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$$|e_i| = |x_i - f_0^{i-1}| \leq \mu_i |\eta|^{\frac{n-i+1}{n}}$$

  - For $d = 0$: $\exists \mu_{i0}(k_i)$, s.t.

$$|e_i| = |x_i - f_0^{i-1}| \leq \mu_{i0} |\eta|$$

- Gain selection: From down upwards $(k_n, k_{n-1}, \cdots, k_1)$ independent of the order. Calculation by maximization of a homogeneous function.
- Parameter $\lambda > 1$ accelerates convergence and increases the allowed bound L, but it also increases the noise effect.
- Convergence Time Estimation

$$\dot{V} \leq -\kappa V(z)^{\frac{p-1}{p}}\,, \kappa > 0\,,\, T(z_0) \leq \frac{p}{\kappa} V^{\frac{1}{p}}(z_0)\,.$$

Outline

First Order differentiator ($n = 2$)

$$
\begin{aligned}
\dot{x}_1 &= -k_1 \lceil x_1 - f \rfloor^{\frac{1}{1-d}} + x_2 \\
\dot{x}_2 &= -k_2 \lceil x_1 - f \rfloor^{\frac{1+d}{1-d}} ,
\end{aligned}
$$

- $x_2(t) \approx f_0^{(1)}$.
- Homogeneous of degree is $-1 \leq d \leq 0$, weights $r_1 = 1 - d$, $r_2 = 1$, and $r_3 = 1 + d$.
- For $d = -1$ Levant's robust and exact differentiator, for $d = 0$ linear differentiator.

First Order differentiator ($n = 2$) contd. I

Lyapunov Function is

$$V\left(z_1,\, z_2\right) \;=\; \frac{1-d}{2-d}\left|z_1\right|^{\frac{2-d}{1-d}} - z_1 z_2 + \left(\frac{\beta+1}{2-d}\right)\left|z_2\right|^{2-d}\,,$$

Derivative $\dot{V}$

$$\begin{aligned}\dot{V} &= -k_1\left|\sigma_1\right|^2 + \tilde{k}_2\left(1+\beta\right)s_1\left\lceil z_1\right\rfloor^{\frac{1+d}{1-d}} - \tilde{k}_2\beta\left|z_1\right|^{\frac{2}{1-d}} \\ \sigma_1 &= \left(\left\lceil z_1\right\rfloor^{\frac{1}{1-d}} - z_2\right)\,,\; s_1 = \left(z_1 - \left\lceil z_2\right\rfloor^{1-d}\right)\end{aligned}$$

First Order differentiator ($n = 2$) contd. II

Required value of k_1 satisfies

$$\frac{k_1}{\tilde{k}_2} = \frac{k_1^2}{k_2} > \omega_2 \triangleq \max_{z \in \mathbb{R}^2} g_2(z_1, z_2) \ ,$$

$$g_2(z_1, z_2) \quad \triangleq \quad \frac{\left(s_1 - \beta \lceil z_2 \rfloor^{1-d}\right) \lceil z_1 \rfloor^{\frac{1+d}{1-d}}}{|\sigma_1|^2} \, .$$

$g_2(z_1, z_2)$ homogeneous of degree zero, upper semicontinuous and has a maximum, achieved on the homogeneous sphere.

Second Order differentiator ($n = 3$)

$$\begin{aligned}
\dot{x}_1 &= -k_1 \lceil x_1 - f \rfloor^{\frac{1-d}{1-2d}} + x_2 \\
\dot{x}_2 &= -k_2 \lceil x_1 - f \rfloor^{\frac{1}{1-2d}} + x_3 \\
\dot{x}_3 &= -k_3 \lceil x_1 - f \rfloor^{\frac{1+d}{1-2d}}
\end{aligned}$$

$x_2(t) \approx f^{(1)}(t)$ and $x_3(t) \approx f^{(2)}(t)$.

$$V(z) = Z_1(z_1, z_2) + \beta_2 Z_2(z_2, z_3) + \frac{\beta_3}{p} |z_3|^p$$

$$\frac{k_2^2}{k_1 k_3} > \omega_{23}\,, \; \frac{k_1 k_2}{k_3} > \omega_{13}\,,$$

Simulations

- Signal $f_0(t) = 0.5\sin(0.5t) + 0.5\cos(t)$,
- Bounded derivative $\left|f_0^{(3)}(t)\right| \leq 1$.
- Simulations for linear $d = 0$, homogeneous $d = -0.5$ and Levant's $d = -1$ differentiators.
- Noise $v(t) = \varepsilon\sin(\omega t)$, $\varepsilon = 0.001$, and $\omega = 1000$.
- Gains $k_1 = 3$, $k_2 = 1.5\sqrt{3}$, $k_3 = 1.1$.
- Euler-method with step size $\tau = 3 \times 10^{-4}$.

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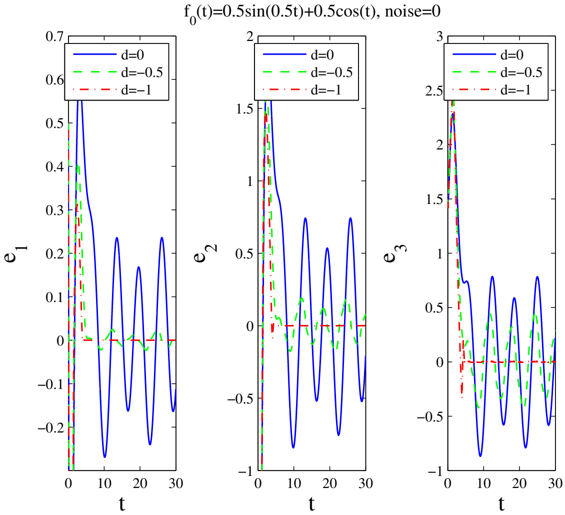
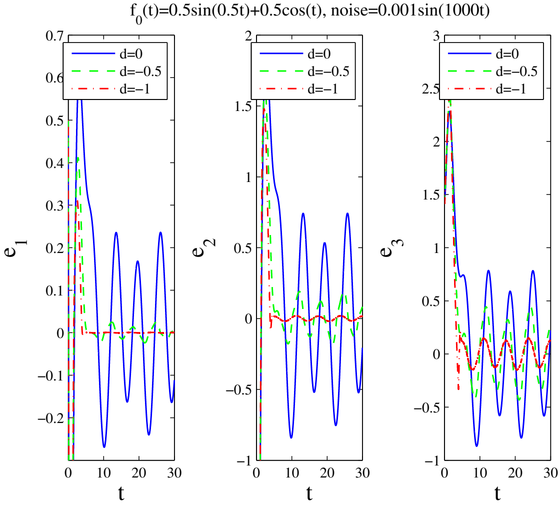
Simulations: No noise



Simulations: Noisy measurement



Overview

Conclusions

- A family of Homogeneous continuous and discontinuous differentiators is proposed.
- Unified family of differentiable LFs is given.
- It allows to
    - Gain calculation (also use of SoS-like methods). But the set of stabilizing gains is not covered!
    - Coefficients for the noise and High derivative effect can be calculated (conservative!).
    - Comparison is possible (future work).
    - Convergence time estimation.
- The discontinuous differentiator is the only capable of exactness.
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Part IV

Construction of Lyapunov Functions using Generalized Forms

Outline

Overview

Generalized Forms (GF) method [Sanchez and Moreno 2014, 2016]

Basic Idea:

Transform a PDE

$$\frac{\partial V(x)}{\partial x} f(x) = -W(x)$$

$\Rightarrow$ Algebraic equation + Positive Definiteness

Motivation: Lyapunov functions for LTI systems

$$\begin{aligned} \text{System:} \quad & \dot{x} = Ax\,, \quad x \in \mathbb{R}^n\,, \\ \text{LF Candidate:} \quad & V(x) = x^T P x\,, \\ \text{LF Derivative:} \quad & -\dot{V} = W(x) = x^T Q x\,, \\ \text{Algebraic Lyapunov Equation:} \quad & PA + A^T P = -Q\,. \end{aligned}$$

Classic forms:

Homogeneous polynomial of degree $m \in \mathbb{Z}_{\geq 0}$

$$f(x) \;=\; \sum_{j=1}^{M} \alpha_j \prod_{i=1}^{n} x_i^{\rho_{i,j}}\,, \quad \rho_{i,j} \in \mathbb{Z}_{\geq 0}\,, \quad \sum_{i=1}^{n} \rho_{i,j} = m$$

Finite $M \in \mathbb{Z}_{>0}$, $\alpha_j \in \mathbb{R}$, $x \in \mathbb{R}^n$.

Example

$$f(x) \;=\; 2x_1^4 + 3x_1^3 x_2 + 5x_1^2 x_2^2 - 6x_1 x_2^3 + 8x_2^4\,.$$

Generalized forms (GF):

Homogeneous function of degree $m \in \mathbb{R}_{\geq 0}$ with weights $\mathbf{r}$

$$f(x) \;=\; \sum_{j=1}^{N} \alpha_j \prod_{i=1}^{n} \upsilon_{i,j}(x_i, \rho_{i,j})\,, \quad \rho_{i,j} \in \mathbb{R}_{\geq 0}\,, \quad \sum_{i=1}^{n} r_i \rho_{i,j} = m$$

$$\upsilon_{i,j}(x_i, \rho_{i,j}) = |x_i|^{\rho_{i,j}},\ \lceil x_i \rfloor^{\rho_{i,j}}$$

Finite $N \in \mathbb{Z}_{>0}$, $\alpha_j \in \mathbb{R}$, $x \in \mathbb{R}^n$.

Example

$$f(x) = \kappa_1 |x_1|^{\frac{5\pi}{2}} + \kappa_2 \lceil x_1 \rfloor^{\frac{\pi}{2}} |x_2|^{\frac{4\pi}{3}}\,, \quad \kappa_i \in \mathbb{R}\,.$$

$m = 5\pi$, $\mathbf{r} = [2, 3]^\top$.

Classic forms $\subset$ GFs

Motivational example

Homogeneous polynomial system ($\kappa = 2$, $\mathbf{r} = [1,3]^\top$)

$$\Sigma : \quad \dot{x}_1 = -x_1^3 + x_2\,, \quad \dot{x}_2 = -x_1^5\,,$$

Weak Lyapunov function [Bacciotti & Rosier, 2005]

$$V(x) = \tfrac{1}{6}x_1^6 + \tfrac{1}{2}x_2^2\,, \quad \dot{V} = -x_1^8\,,$$

Theorem [Sanchez, 2016]

For Σ, there is no strict LF in the class of homogeneous polynomials of any degree m for any weights $\mathbf{r}$.

Strict Lyapunov function for Σ

$$V(x) = \alpha_1 x_1^6 - \alpha_{12} x_1 \lceil x_2 \rfloor^{\frac{5}{3}} + \alpha_2 x_2^2\,, \quad \text{(GF!)}$$

GF systems: HOSM

Twisting algorithm [Levant, 1993]

$$\dot{x}_1 = x_2\,, \quad \dot{x}_2 = -k_1 \lceil x_1 \rfloor^0 - k_2 \lceil x_2 \rfloor^0\,,$$

Super–Twisting algorithm [Levant, 1993]

$$\dot{x}_1 = -k_1 \lceil x_1 \rfloor^{\frac{1}{2}} + x_2\,, \quad \dot{x}_2 = -k_2 \lceil x_1 \rfloor^0\,,$$

CTA [Torres et al., 2013]

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -k_1 \lceil x_1 \rfloor^{\frac{1}{3}} - k_2 \lceil x_2 \rfloor^{\frac{1}{2}} + x_3 \\ \dot{x}_3 &= -k_3 \lceil x_1 \rfloor^0 - k_4 \lceil x_2 \rfloor^0 \end{aligned}$$

$$\lceil \cdot \rfloor^\rho = \mathrm{sign}(\cdot)|\cdot|^\rho$$

More GF systems

Continuous homogeneous systems (Finite time)

$$\dot{x}_1 = x_2\,, \quad \dot{x}_2 = -k_1 \lceil x_1 \rfloor^{\frac{1}{2}} - k_2 \lceil x_2 \rfloor^{\frac{2}{3}}\,, \quad \kappa = -1\,, \quad \mathbf{r} = [3, 2]^\top$$

Polynomial homogeneous systems

$$\dot{x}_1 = -x_1^3 + x_2\,, \quad \dot{x}_2 = -x_1^5\,, \quad \kappa = 2\,, \quad \mathbf{r} = [1, 3]^\top$$

Linear systems

$$\dot{x} = Ax\,, \quad x \in \mathbb{R}^n\,, \quad \kappa = 0\,, \quad \mathbf{r} = [1, \ldots, 1]^\top$$

Overview

GFs properties

Theorem

1. Sums of GFs (degree m, weights $\mathbf{r}$) are GFs of degree m.
2. F_a and F_b GF of degree m_a and m_b, with weights $\mathbf{r}$, $F_a F_b$ is a GF of degree $m_a + m_b$.

Theorem

1. A GF is differentiable almost everywhere (coordinate hyperplanes).
2. A continuous GF is differentiable everywhere if its exponents $\rho_{i,j} \neq 0$ are such that

$$\begin{array}{l} \rho_{i,j} \geq 1\,, \text{ if } \upsilon_i(x_i, \rho_{i,j}) = \lceil x_i \rfloor^{\rho_{i,j}} \\ \rho_{i,j} > 1\,, \text{ if } \upsilon_i(x_i, \rho_{i,j}) = |x_i|^{\rho_{i,j}} \end{array}\,, \ \forall i, j\,.$$

3. Partial derivatives of a differentiable GF are GFs.

GFs properties

Corollary

- $\dot{x} = f(x)$, $x \in \mathbb{R}^n$, is a GF system of degree κ with $\mathbf{r}$.
- $V : \mathbb{R}^n \to \mathbb{R}$ is a GF of degree m with weights $\mathbf{r}$.

$\Rightarrow W(x) = -\nabla V(x) \cdot f(x)$ is GF of degree $\bar{m} = m + \kappa$.

Structure for positive definiteness

$$\begin{aligned} V(x, \alpha) &= \sum_{i=1}^{n} \alpha_i |x_i|^{\frac{m}{r_i}} + \sum_{j=1}^{q} \bar{\alpha}_j \prod_{i=1}^{n} \upsilon_{i,j}(x_i, \rho_{i,j}), \\ W(x, \beta) &= \sum_{i=1}^{n} \beta_i |x_i|^{\frac{\bar{m}}{r_i}} + \sum_{j=1}^{\bar{q}} \bar{\beta}_j \prod_{i=1}^{n} \bar{\upsilon}_{i,j}(x_i, \bar{\rho}_{i,j}), \end{aligned}$$

Polynomial characterization

Commensurable exponents: $\rho_i/\rho_j \in \mathbb{Q}$

Isomorphism: $d^\gamma : \mathcal{P}_n \to \mathcal{D}_\gamma$

$$d^\gamma(y) = [\sigma_1 y_1^{\mu_1}, \ldots, \sigma_n y_n^{\mu_n}]^\top, \; \mu_i \in \mathbb{Q}_{>0}$$

Hyperoctants: $\mathcal{D}_\gamma \subset \mathbb{R}^n$. $\sigma_i = \operatorname{sign}(x_i)$, $x \in \mathcal{D}_\gamma$
$\mathcal{P}_n = \{z \in \mathbb{R}^n \,|\, z_i > 0, i = 1, 2, \ldots, n\}$.

Lemma

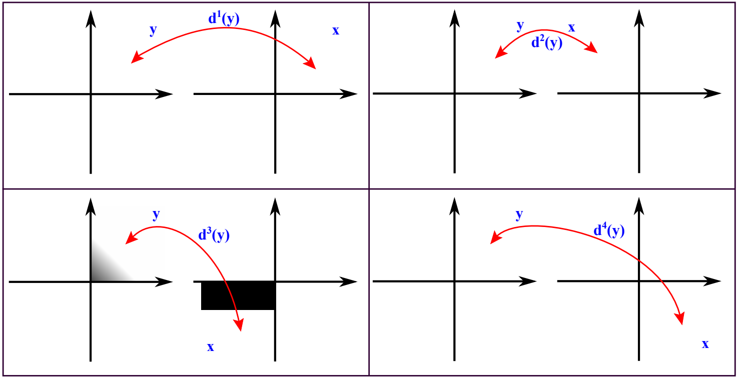
If $f : \mathbb{R}^n \to \mathbb{R}$ is GF of degree m with weights $\mathbf{r}$ and rational exponents, then there exist $\mu_i \in \mathbb{Q}_{>0}$ such that every $f_{\mathcal{D}_\gamma} \circ d^\gamma : \mathcal{P}_n \to \mathbb{R}$ is a form.

Associated forms of a GF f

$$f(x) : \{f_1(y), \ldots, f_{2^n}(y)\}, \quad f_i : \bar{\mathcal{P}}_n \to \mathbb{R}$$

Polynomial characterization

Isomorphism:



Lemma

A GF $f : \mathbb{R}^n \to \mathbb{R}$ is positive definite if its associated forms are positive definite.

Example

GF

$$V(x) = |x_1|^{\frac{5}{3}} + x_1 x_2 + |x_2|^{\frac{5}{2}}\,, \quad m = 5\,,\ \mathbf{r} = [3, 2]^\top$$

Isomorphism

$$d^\gamma(z) = [\sigma_1 z_1^3, \sigma_2 z_2^2]^\top$$

$V_\gamma = V \circ d^\gamma : \bar{\mathcal{P}} \to \mathbb{R}$

- $\bar{\mathcal{D}}_1 = \{x_1 \geq 0, x_2 \geq 0\}$, $V_1(z) = z_1^5 + z_1^3 z_2^2 + z_2^5$
- $\bar{\mathcal{D}}_2 = \{x_1 \leq 0, x_2 \geq 0\}$, $V_2(z) = z_1^5 - z_1^3 z_2^2 + z_2^5$
- $\bar{\mathcal{D}}_3 = \{x_1 \leq 0, x_2 \leq 0\}$, $V_3(z) = z_1^5 + z_1^3 z_2^2 + z_2^5$
- $\bar{\mathcal{D}}_4 = \{x_1 \geq 0, x_2 \leq 0\}$, $V_4(z) = z_1^5 - z_1^3 z_2^2 + z_2^5$

$$V(x) : \{V_1(z), V_2(z), V_3(z), V_4(z)\}$$

Overview

Pólya's Theorem

Theorem (Pólya, 1928)

The (classic) Form $F : \bar{\mathcal{P}}_n \setminus \{0\} \to \mathbb{R}$*, is positive if and only if there exists* $p_0 \in \mathrm{N}$ *such that for all* $p \geq p_0$*, the coefficients of the form*

$$G(z) = (z_1 + z_2 + \cdots + z_n)^p F(z), \quad \forall z \in \bar{\mathcal{P}}_n \setminus \{0\},$$

are positive.

Example

$$V(z) = \alpha_1 z_1^3 - \alpha_2 z_1^2 z_2 + \alpha_3 z_2^3 \,, \quad \alpha_i > 0 \,,$$

$$G_p(z) = (z_1 + z_2)^p \, V(z) \,,$$

Pólya's Theorem

$p = 1$

$$G_1(z) = \alpha_1 z_1^4 + (\alpha_1 - \alpha_2) z_1^3 z_2 - \alpha_2 z_1^2 z_2^2 + \alpha_3 z_1 z_2^3 + \alpha_3 z_2^4 .$$

$p = 2$

$$\begin{aligned} G_2(z) &= \alpha_1 z_1^5 + (2\alpha_1 - \alpha_2) z_1^4 z_2 + (\alpha_3 - \alpha_2) z_1^2 z_2^3 \\ &\quad + (\alpha_1 - 2\alpha_2) z_1^3 z_2^2 + 2\alpha_3 z_1 z_2^4 + \alpha_3 z_2^5 . \end{aligned}$$

Inequalities

$$\alpha_1 > 0,\ 2\alpha_1 - \alpha_2 \geq 0,\ \alpha_3 - \alpha_2 \geq 0,\ \alpha_1 - 2\alpha_2 \geq 0,\ \alpha_3 > 0\,,$$

Pólya's Theorem

System of linear inequalities $A_v\alpha \succ 0$

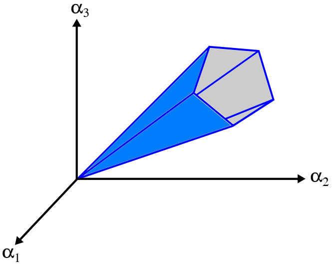
$$A_v = \begin{bmatrix} 1 & 0 & 0 & 2 & 0 & 1 \\ 0 & 1 & 0 & -1 & -1 & -2 \\ 0 & 0 & 1 & 0 & 1 & 0 \end{bmatrix}^\top, \quad \alpha = [\alpha_1 \; \alpha_2 \; \alpha_3]^\top.$$

Polyhedral cone

$$C = \{\alpha \in \mathbb{R}^d \; : \; A\alpha \succeq 0\}$$

Minkowski-Weyl

$$C = \{\alpha = B\gamma \; : \; 0 \preceq \gamma \in \mathbb{R}^q\}$$



A: Faces, B: Edges

Software: Skeleton [Zolotykh, 2012]

SOS representation

Sum of Squares (SOS) representation: Hilbert's 17th prob.

The (classic) Form F of degree $2q$ is positive semi-definite if

$$F(z) = \sum_{i=1}^{N} (f_i)^2 \,.$$

Example

$$F(z) = z_1^2 + 2z_1 z_2 + z_2^2 = (z_1 + z_2)^2 \,,$$

Example

$$\begin{aligned} F(z) =& z_1^6 - 2z_1^4 z_2 z_3 + z_1^2 z_2^4 + z_1^2 z_2^2 z_3^2 - 2z_1 z_2^2 z_3^3 + z_3^6 \\ =& (z_1^3 - z_1 z_2 z_3)^2 + (z_3^3 - z_1 z_2^2)^2 \,. \end{aligned}$$

SOS representation

SOS-Quadratic form [Choi et al., 1995]

$$F(z) : SOS \iff F(z) = y(z)^T P y(z)\,,$$

$$F = F(z;\alpha)\text{ , LMI problem: } \quad P(\alpha) \geq 0$$

Software: SOSTOOLS [Prajna et al., 2002-2005] .

Positive definiteness

$$\bar{F}(z) = F(z) - \epsilon \sum_{i=1}^{n} z_i^m\,, \quad \epsilon \in \mathbb{R}_{>0}$$

Positive definite GFs

Let $F = F(x;\alpha)$ be a GF and $\{F_i\}$ its associated forms

Pólya's Theorem

$$G_p(z;\alpha) = (z_1 + z_2 + \cdots + z_n)^p F_i(z;\alpha)\,,$$

$$\text{Linear inequalities: } A_i\alpha \succ 0\,,$$

SOS representation

$$\bar{F}_i(z;\alpha) = F_i(z;\alpha) - \epsilon \sum_{i=1}^{n} z_i^m\,, \quad \epsilon \in \mathbb{R}_{>0}$$

$$\text{LMIs: } P_i(\alpha) \geq 0\,.$$

Adequate isomorphism.

Comments

Pólya

- *Necessary and Sufficient* condition for positive definiteness
- Leads to Linear Inequalities (not LMIs)
- The complete solution for a given power p can be completely characterized
- Available Software (e.g., Skeleton)

SOS

- A *Sufficient* condition for positive definiteness
- Leads to LMIs
- Available Software (e.g., SOSTOOLS)
- Allows to include objective functions (optimization)

Overview

GF Lyapunov function

GF system:

$$\dot{x} = f(x;k)$$

GF LF candidate:

$$V(x;\alpha)$$

Derivative:

$$W(x;\beta)\,,$$

Associated forms

$$\{V_i(z;\alpha),\, W_i(z;\beta)\}$$

β bilinear

$$\beta = \beta(\alpha,k)\,, \quad \beta = M(k)\alpha\,, \quad \beta = \bar{M}(\alpha)k$$

Algorithm. I

Given a GF system $\dot{x} = f(x;k)$ of degree κ with weights $\mathbf{r}$,

Step 1 Chose some terms $\upsilon_{ij}(x_i, \rho_i)$ in

$$V(x,\alpha) \;=\; \sum_{i=1}^{n} \alpha_i |x_i|^{\frac{m}{r_i}} + \sum_{j=1}^{q} \bar{\alpha}_j \prod_{i=1}^{n} \upsilon_{i,j}(x_i, \rho_{i,j})$$

Step 2 Take the derivative of V along the trajectories of the system and obtain

$$W(x,\beta) \;=\; \sum_{i=1}^{n} \beta_i |x_i|^{\frac{\bar{m}}{r_i}} + \sum_{j=1}^{\bar{q}} \bar{\beta}_j \prod_{i=1}^{n} \bar{\upsilon}_{i,j}(x_i, \bar{\rho}_{i,j})$$

Algorithm. II

Step 3 Considering

- Homogeneity: $\sum_{i=1}^{n} r_i \rho_{i,j} = m$
- Differentiability: $\begin{cases} m > \max_i\{r_i\} \\ \rho_{i,j} \geq 1\,, \text{ for } \upsilon_i(x_i, \rho_{i,j}) = \lceil x_i \rfloor^{\rho_{i,j}} \\ \rho_{i,j} > 1\,, \text{ for } \upsilon_i(x_i, \rho_{i,j}) = |x_i|^{\rho_{i,j}} \end{cases}$

restrict the exponents $\rho_{i,j}$ and the signs of $\bar{\alpha}_j$ such that the coefficients β_i can be strictly positive. If not, go back to **Step 1** and increase q or change $\upsilon_{ij}(x_i, \rho_i)$.

Step 4 Set m and $\rho_{i,j}$.

Step 5 Chose μ_i in: $d^{\gamma}(y) = [\sigma_1 y_1^{\mu_1}, \ldots, \sigma_n y_n^{\mu_n}]^\top$

Algorithm. III

Step 6 Compute the associated forms

$$\{V_1, \ldots, V_{2^n}\}, \quad \{W_1, \ldots, W_{2^n}\}$$

Step 7 Find α and k for positive definiteness of V_i, W_i

- Solving Pólya's inequalities
- Finding SOS representation

Bilinear problem!

Analysis (k given):

Pólya's procedure

$$\{A_{V_i}\alpha \succ 0,\, A_{W_i}\beta \succ 0\}\,, \quad \beta = M(k)\alpha$$

Solve for α the system of linear inequalities:

$$\{A_{V_i}\alpha \succ 0,\, A_{W_i}M(k)\alpha \succ 0\}$$

SOS procedure

Define the forms:

$$\bar{V}_j(y) = V_j(y) - \epsilon\sum_{i=1}^{n} y_i^{\delta}\,, \quad \bar{W}_j(y) = W_j(y) - \epsilon\sum_{i=1}^{n} y_i^{\bar{\delta}}\,,$$

δ, $\bar{\delta}$, degrees of V_i, W_i. Solve for α the system of LMIs:

$$\{P_{\bar{V}_i}(\alpha) \geq 0,\, P_{\bar{W}_i}(\alpha) \geq 0\}$$

Design:

Pólya's procedure

$$\{A_{V_i}\alpha \succ 0,\ A_{W_i}\beta \succ 0\}, \quad \beta = \bar{M}(\alpha)k$$

- Solve for α the system $\{A_{V_i}\alpha \succ 0\}$ and choose an α^*
- Solve for k the system $\{A_{W_i}\bar{M}(\alpha^*)k \succ 0\}$

SOS procedure

$$\{P_{\bar{V}_i}(\alpha) \geq 0,\ P_{\bar{W}_i}(\alpha, k) \geq 0\}$$

- Solve for α the LMIs $\{P_{\bar{V}_i}(\alpha) \geq 0\}$ and choose an α^*
- Solve for k the LMIs $\{P_{\bar{W}_i}(\alpha^*, k) \geq 0\}$

Overview

Super–Twisting I

Super–Twisting algorithm [Levant, 1993]

$$\dot{x}_1 = -k_1 \lceil x_1 \rfloor^{\frac{1}{2}} + x_2 \,, \quad \dot{x}_2 = -k_2 \lceil x_1 \rfloor^{0} \,,$$

Homogeneous of degree $\kappa = -1$ with weights $\mathbf{r} = [2, 1]^\top$.

Lyapunov function candidate

$$V(x) = \alpha_1 |x_1|^{\frac{m}{2}} + \bar{\alpha}_2 \lceil x_1 \rfloor^{\rho_1} \lceil x_2 \rfloor^{\rho_2} + \alpha_3 |x_2|^{m}.$$

Homogeneity $\rho_2 = m - 2\rho_1$.
Necessary conditions for positive definiteness: $\alpha_1, \alpha_3 > 0$.
Differentiability: $m > 2$, $\rho_1 \geq 1$ and $\rho_2 = m - 2\rho_1 \geq 1$.

Choosing $m = 3$

$$V(x) = \alpha_1 |x_1|^{\frac{3}{2}} + \bar{\alpha}_2 \lceil x_1 \rfloor^{\rho_1} \lceil x_2 \rfloor^{3-2\rho_1} + \alpha_3 |x_2|^{3}.$$

Super–Twisting II

$\dot{V} = -W(x)$

$$W(x) = \frac{3\alpha_1 k_1}{2}|x_1| - \frac{3\alpha_1}{2}\lceil x_1 \rfloor^{\frac{1}{2}} x_2 + 3\alpha_3 k_2 \lceil x_2 \rfloor^2 \lceil x_1 \rfloor^0 + \bar{\alpha}_2 k_2 (3 - 2\rho_1)|x_1|^{\rho_1}|x_2|^{2-2\rho_1} + \bar{\alpha}_2 k_2 \rho_1 \lceil x_1 \rfloor^{\rho_1 - \frac{1}{2}} \lceil x_2 \rfloor^{3-2\rho_1} - \bar{\alpha}_2 \rho_1 |x_1|^{\rho_1 - 1}|x_2|^{2-2\rho_1}.$$

$\rho_1 = 1$ and $-\bar{\alpha}_2 = \alpha_2 > 0$.

LF Candidate

$$V(x) = \alpha_1 |x_1|^{\frac{3}{2}} - \alpha_2 x_1 x_2 + \alpha_3 |x_2|^3,$$

$$W(x) = \beta_1 |x_1| - \beta_2 \lceil x_1 \rfloor^{\frac{1}{2}} x_2 + \beta_3 |x_2|^2 + \beta_4 \lceil x_1 \rfloor^0 |x_2|^2,$$

Super–Twisting III

Coefficients of the Derivative

$\beta_1 = \frac{3}{2}\alpha_1 k_1 - \alpha_2 k_2, \quad \beta_2 = \frac{3}{2}\alpha_1 + \alpha_2 k_1, \quad \beta_3 = \alpha_2, \quad \beta_4 = 3\alpha_3 k_2$

Note: β_i is linear in α_j and linear in k_j but not in both.

LF conditions

Find α_i, k_i so that $V > 0$ and $W > 0$.

Isomorphism

$$d^{\gamma}(z) = [\sigma_1 z_1^2, \sigma_2 z_2]^{\top}$$

Super–Twisting IV

$V_\gamma = V \circ d^\gamma : \bar{\mathcal{P}} \to \mathbb{R}$

- $\bar{\mathcal{D}}_1 = \{x_1 \geq 0, x_2 \geq 0\}, \;\; V_1(z) = \alpha_1 z_1^3 - \alpha_2 z_1^2 z_2 + \alpha_3 z_2^2$
- $\bar{\mathcal{D}}_2 = \{x_1 \leq 0, x_2 \geq 0\}, \;\; V_2(z) = \alpha_1 z_1^3 + \alpha_2 z_1^2 z_2 + \alpha_3 z_2^2$
- $\bar{\mathcal{D}}_3 = \{x_1 \leq 0, x_2 \leq 0\}, \;\; V_3(z) = \alpha_1 z_1^3 + \alpha_2 z_1^2 z_2 + \alpha_3 z_2^2$
- $\bar{\mathcal{D}}_4 = \{x_1 \geq 0, x_2 \leq 0\}, \;\; V_4(z) = \alpha_1 z_1^3 - \alpha_2 z_1^2 z_2 + \alpha_3 z_2^2$

$W_\gamma = W \circ d^\gamma : \bar{\mathcal{P}} \to \mathbb{R}$

- $\bar{\mathcal{D}}_1 = \{x_1 \geq 0, x_2 \geq 0\}, W_1(z) = \beta_1 z_1^2 - \beta_2 z_1 z_2 + (\beta_3 + \beta_4) z_2^2$
- $\bar{\mathcal{D}}_2 = \{x_1 \leq 0, x_2 \geq 0\}, W_2(z) = \beta_1 z_1^2 + \beta_2 z_1 z_2 + (\beta_3 - \beta_4) z_2^2$
- $\bar{\mathcal{D}}_3 = \{x_1 \leq 0, x_2 \leq 0\}, W_3(z) = \beta_1 z_1^2 + \beta_2 z_1 z_2 + (\beta_3 - \beta_4) z_2^2$
- $\bar{\mathcal{D}}_4 = \{x_1 \geq 0, x_2 \leq 0\}, W_4(z) = \beta_1 z_1^2 - \beta_2 z_1 z_2 + (\beta_3 + \beta_4) z_2^2$

Super–Twisting V

$\{x_1 x_2 < 0\}$: $z_1 \geq 0,\ z_2 \geq 0$

$$\begin{aligned} V(z) &= \alpha_1 z_1^3 + \alpha_2 z_1^2 z_2 + \alpha_3 z_2^3 \,, \\ W(z) &= \beta_1 z_1^2 + \beta_2 z_1 z_2 + (\beta_3 - \beta_4) z_2^2 \,. \end{aligned}$$

$\beta_3 > \beta_4$.

$\{x_1 x_2 \geq 0\}$: $z_1 \geq 0,\ z_2 \geq 0$

$$\begin{aligned} V(z) &= \alpha_1 z_1^3 - \alpha_2 z_1^2 z_2 + \alpha_3 z_2^3 \,, \\ W(z) &= \beta_1 z_1^2 - \beta_2 z_1 z_2 + (\beta_3 + \beta_4) z_2^2 . \end{aligned}$$

Just these forms must be analysed!
(V was analysed in the example of Pólya's theorem)

Super-Twisting, Pólya's procedure

Fix $\alpha = [2.1,\ 1,\ 1.1]^\top$,

$$G_2(z) = (z_1 + z_2)^p\, W(z) \quad \Rightarrow \quad A_w \beta \succ 0\,,$$

Double description

$$A_w M(\alpha)[1\ k^\top]^\top > 0 \quad \Leftrightarrow \quad k = B_w \gamma\,,$$

$\gamma \in \mathbb{R}^q$, $\gamma_i > 0$, $\sum_{i=1}^q \gamma_i = 1$, q is the number of columns of B_w.

$$\text{Solution for } p = 6 B_w = \begin{bmatrix} 3.788 & 2.325 & 3.019 \\ 0.303 & 0.303 & 0.257 \end{bmatrix}\,,$$

For example, with $\gamma = (1/3)[1,\ 1,\ 1]^\top$

$$k_1 = 3.04,\ \ k_2 = 0.28$$

Super-Twisting, SOS procedure I

Change of variables: $z_1 > 0,\ z_2 > 0,\ y_1, y_2 \in \mathbb{R}$

$$(z_1, z_2) \mapsto (y_1^2, y_2^2).$$

Classical Forms of even degree: $y \in \mathbb{R}^2$

$$V(y) = \alpha_1 y_1^6 - \alpha_2 y_1^4 y_2^2 + \alpha_3 y_2^6\,,$$

$$W(y) = \beta_1 y_1^4 - \beta_2 y_1^2 y_2^2 + (\beta_3 + \beta_4) y_2^4\,.$$

SOS $\Rightarrow$ LMI [Parrilo, 2000]

$$\bar{V}(y) = V(y) - \epsilon(y_1^6 + y_2^6) > 0\,, \quad \epsilon > 0\,.$$

$$\bar{V}(y) = \psi^T(y) Q_v \psi(y)\,, \quad \psi(y) = [y_1^3,\ y_1^2 y_2,\ y_1 y_2^2,\ y_2^3]^T\,.$$

Super-Twisting, SOS procedure II

$$Q_v = \begin{bmatrix} \bar{\alpha_1} & 0 & -\lambda_1 & 0 \\ 0 & 2\lambda_1\alpha_1\alpha_0 - \alpha_2 & 0 & -\lambda_2 \\ -\lambda_1 & 0 & \alpha_0 - 2\lambda_2\alpha_2\bar{\alpha_3} & 0 \\ 0 & -\lambda_2 & 0 & \bar{\alpha_3} \end{bmatrix} > 0\,.$$

Overview

HOSM Differentiator

$$f(t) = f_0(t) + \nu(t), \quad \left| f_0^{(n)}(t) \right| \leq L$$

Levant's Differentiator

$$\begin{aligned} \dot{x}_i &= -k_i \lceil x_1 - f \rfloor^{\frac{n-i}{n}} + x_{i+1}, i = 1, \cdots, n-1 \\ \dot{x}_n &= -k_n \lceil x_1 - f \rfloor^{0} . \end{aligned}$$

Dynamics of the Differentiation error: $z_i = \frac{x_i - f_0^{(i-1)}}{k_{i-1}}$

$$\begin{aligned} \dot{z}_i &= -\tilde{k}_i \left(\lceil z_1 + \nu \rfloor^{\frac{n-i}{n}} - z_{i+1} \right), \quad \tilde{k}_i = \frac{k_i}{k_{i-1}}, \\ \dot{z}_n &= -\tilde{k}_n \lceil z_1 + \nu \rfloor^{0} - \frac{f_0^{(n)}(t)}{k_{n-1}} . \end{aligned}$$

Homogeneous: degree $d = -1$, weights $\mathbf{r} = (n, n-1, \cdots, 1)$.

Generalized Form as Lyapunov Function

LF: for $p \geq 2n-1$ and any $\beta_i > 0$

$$V(z) = \sum_{j=1}^{n-1} \beta_j Z_j(z_j,\, z_{j+1}) + \beta_n \frac{1}{p} |z_n|^p ,\; \beta_i > 0$$

$$Z_i(z_i,\, z_{i+1}) = \tfrac{n+1-i}{p} \left|z_i\right|^{\frac{p}{n+1-i}} + \\ - z_i \left\lceil z_{i+1} \right\rfloor^{\frac{p-n-1+i}{n-i}} + \left(\tfrac{p-n-1+i}{p}\right) \left|z_{i+1}\right|^{\frac{p}{n-i}} .$$

Convergence Time Estimation

$$\dot{V} \leq -\kappa V(z)^{\frac{p-1}{p}} ,\kappa > 0$$

$$T(z_0) \leq \frac{p}{\kappa} V^{\frac{1}{p}}(z_0) .$$

Gains calculation by SOS

Gain Calculation using SOS and $p = 2n - 1$

n	k_1	k_2	k_3	k_4	L
2	2.12	1.02	–	–	1
3	3.01	4.95	1.03	–	1
4	5.81	17.75	15.45	1.02	1

Summary

Pros

- Provides a computable way to calculate LFs for a fairly general class of homogeneous systems.
- It can be extended to non homogeneous systems.
- The algebraic problem to solve is a system of linear inequalities (Pólya) or an LMI (SOS). It is linear in the coefficients of the LF candidate and in the gains.

Cons

- Restricted to "polynomial" systems
- Course of high p for Pólya and SOS.

Summary

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Part V

Continuous HOSM Controllers

Outline

Overview

Overview

Discontinuous HOSM Controller

Perturbed second-order plant

$$\begin{aligned}\dot{x}_1 &= x_2 \\ \dot{x}_2 &= u + \mu(t) \ ,\end{aligned}$$

- Discontinuous controller (SOSM), e.g. Twisting controller, rejects bounded perturbation,
- strong chattering,
- Precision

$$|x_1| \leq \nu_1 \tau^2, \quad |x_2| \leq \nu_2 \tau \ .$$

- Chattering reduction requires continuous control signal.

Chattering Attenuation: Standard

$$
\begin{aligned}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= x_3 \triangleq u + \mu(t) \\
\dot{x}_3 &= \dot{u} + \dot{\mu}(t) \\
\dot{u} &= k_3 \vartheta_2 (x_1,\, x_2,\, x_3)
\end{aligned}
$$

Properties

- Levant 2003
- Continuous control signal $u(t) \Rightarrow$ chattering attenuation
- Rejects Lipschitz continuous (possibly unbounded) perturbation,
- Precision $|x_1| \leq \nu_1 \tau^3, \quad |x_2| \leq \nu_2 \tau^2, \quad |x_3| \leq \nu_3 \tau$
- Drawback: It requires $(x_1,\, x_2)$ and $\dot{x}_2$!

Overview

Preview: Relative degree $r = 1$, the Super-Twisting

- System:

$$\dot{x}_1 = u + \rho(x, t)$$

- Discontinuous Control:

$$u = -k \operatorname{sign}(x_1)$$

- Discontinuous Integral Control (Super-Twisting):

$$\begin{aligned} u &= -k_1 |x_1|^{\frac{1}{2}} \operatorname{sign}(x_1) + z \\ \dot{z} &= -k_2 \operatorname{sign}(x_1) \end{aligned}$$

- Closed Loop System:

$$\begin{aligned} \dot{x}_1 &= -k_1 |x_1|^{\frac{1}{2}} \operatorname{sign}(x_1) + x_2 \\ \dot{x}_2 &= -k_2 \operatorname{sign}(x_1) + \dot{\rho}(x, t) \end{aligned}$$

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Super-Twisting

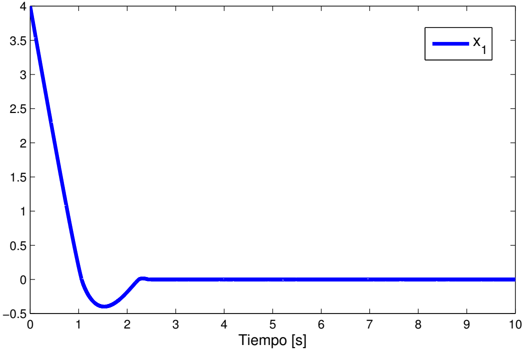

Figure : State Trajectory with $\rho(t) = 0.5\sin(t) + 0.25\sin(2t)$

- Robust stabilization in finite time
- Continuous control signal

Super-Twisting

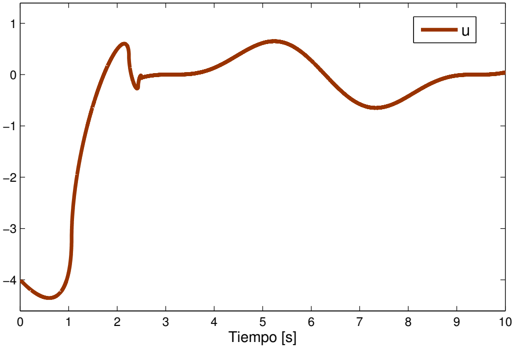

Figure : Super-Twisting Control Signal

Overview

Continuous Terminal Sliding Mode Controller

$$
\begin{aligned}
u &= -k_1 L^{\frac{2}{3}} \lceil \phi_L(x_1, x_2) \rfloor^{\frac{1}{3}} + z \\
\dot{z} &= -k_2 L \lceil \phi_L(x_1, x_2) \rfloor^{0}
\end{aligned}
$$

$$
\phi_L(x_1, x_2) = x_1 + \frac{\alpha}{L^{\frac{1}{2}}} \lceil x_2 \rfloor^{\frac{3}{2}} \,, k_i > 0,\ L > 0\,.
$$

Stability proof: Lyapunov function

$$
V(x) = \beta |x_1|^{\frac{5}{3}} + x_1 x_2 + \frac{2}{5} \alpha |x_2|^{\frac{5}{2}} - \frac{1}{k_1^3} x_2 x_3^3 + \gamma |x_3|^5 \,,
$$

$$
x_3 \triangleq z + \mu
$$

Properties

- Kamal, Moreno, Chalanga, Bandyopadhyay, Fridman (2016).
- Continuous control signal $u(t) \Rightarrow$ chattering attenuation
- It rejects Lipschitz continuous (possibly unbounded) perturbation,
- Precision

$$|x_1| \leq \nu_1 \tau^3, \quad |x_2| \leq \nu_2 \tau^2, \quad |x_3| \leq \nu_3 \tau$$

- Advantage: It only requires (x_1, x_2) and not $\dot{x}_2$!
- Estimation of the perturbation: $z(t) \to \mu(t)$.

Gain calculation by function maximization

Set	1	2	3	4
k_1	4.4	4.5	7.5	16
k_2	2.5	2	2	7
α	20	28.7	7.7	1
Δ	1	1	1	1

Table : Sets of gain values obtained by maximization for $L = 1$.

Phaseportrait: Sliding-like behavior

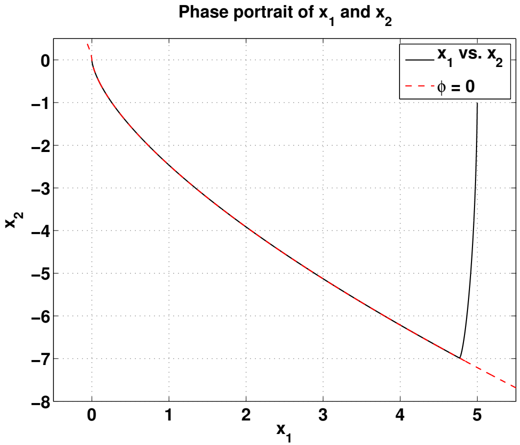

Figure : Phaseportrait

Phaseportrait: Twisting-like behavior

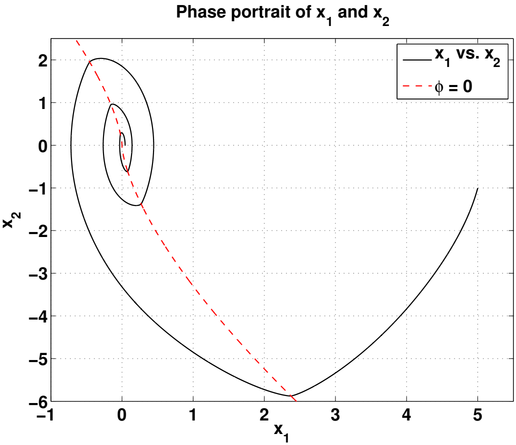

Figure : Phaseportrait

Overview

Continuous Twisting Algorithm

$$\begin{aligned} u(x) &= -k_1 \lceil x_1 \rfloor^{\frac{1}{3}} - k_2 \lceil x_2 \rfloor^{\frac{1}{2}} + z \\ \dot{z} &= -k_3 \lceil x_1 \rfloor^{0} - k_4 \lceil x_2 \rfloor^{0} \end{aligned} ,$$

Stability proof: Lyapunov function

$$V(x) = \alpha_1 |x_1|^{\frac{5}{3}} + \alpha_2 x_1 x_2 + \alpha_3 |x_2|^{\frac{5}{2}} + \alpha_4 x_1 \lceil x_3 \rfloor^{2} - \alpha_5 x_2 x_3^3 + \alpha_6 |x_3|^{5} .$$

$$x_3 \triangleq z + \mu$$

Properties

- Torres, Sanchez, Fridman, Moreno (2015), [25]
- Continuous control signal $u(t) \Rightarrow$ chattering attenuation
- It rejects Lipschitz continuous (possibly unbounded) perturbation,
- Precision

$$|x_1| \leq \nu_1 \tau^3, \quad |x_2| \leq \nu_2 \tau^2, \quad |x_3| \leq \nu_3 \tau$$

- Advantage: It only requires (x_1, x_2) and not $\dot{x}_2$!
- Estimation of the perturbation: $z(t) \to \mu(t)$.
- Gain calculation using Polya's Theorem.
- Convergence Time estimation

$$T_c \leq \frac{5}{\gamma} V^{\frac{1}{5}}(x(0)) \ ,$$

Virtues of Continuous HOSM

- Continuous control signal $\Rightarrow$ chattering attenuation.
- Extension to arbitrary order

$$
\begin{aligned}
\dot{x}_i &= x_{i+1},\ i = 1, ..., \rho - 1, \\
\dot{x}_\rho &= -k_1 \phi(x) + z + \mu(t) \\
\dot{z} &= -k_2 \lceil \phi(x) \rfloor^0
\end{aligned}
$$

- Rejects Lipschitz (possibly unbounded) continuous perturbations versus bounded perturbations of HOSM.
- Requires only x and not $\dot{x}_\rho$.
- Lyapunov approach also extended for arbitrary order systems.
- Interesting approach from (Chitour, Harmouche, Laghrouche).

Overview

Continuous Integral Controller (PID)

- System

$$\begin{aligned}\dot{x}_1 &= x_2 \\ \dot{x}_2 &= u + \rho(t) \ ,\end{aligned}$$

- PID-Controller (e.g. linear)

$$\begin{aligned}u &= -k_1(x_1, x_2) + k_I(x_3) \\ \dot{x}_3 &= -k_2(x_1, x_2) \ ,\end{aligned}$$

- $k_{1,2}(x_1, x_2)$ continuous, $k_I(x_3)$ continuous/discontinuous.
- Constant perturbations/references $\Rightarrow$ Asymptotic convergence and insensitive to perturbation!
- Arbitrary perturbations/ref $\Rightarrow$ Practical convergence.
- Estimation of $\rho(t)$ is not required for implementation.
- More general: Internal Model Principle based controller.

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Proposed Solution

- Combine Integral Action and Discontinuous Control.
- $k_1(x_1, x_2)$ and $k_I(x_3)$ continuous, $k_2(x_1, x_2)$ discontinuous.
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- For simplicity (!?) we add Homogeneity.

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Overview

Outline

Continuous and Homogeneous State Feedback Controller

$$u = -k_1 \lceil x_1 \rfloor^{\frac{1}{3}} - k_2 \lceil x_2 \rfloor^{\frac{1}{2}}$$

Closed Loop System:

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -k_1 \lceil x_1 \rfloor^{\frac{1}{3}} - k_2 \lceil x_2 \rfloor^{\frac{1}{2}} + \rho(t) \ , \end{aligned}$$

Lyapunov Function:

$$V(x_1, x_2, x_3) = \gamma_1 |x_1|^{\frac{5}{3}} + \gamma_{12} x_1 x_2 + |x_2|^{\frac{5}{2}} \ ,$$

Sensitive to perturbations.

Homogeneous Integral + State Feedback Controller

Discontinuous Integral Controller ($k_1, k_2, k_3 > 0,\ k_4 \in \mathbb{R}$)

$$\begin{aligned} u &= -k_1 \lceil x_1 \rfloor^{\frac{1}{3}} - k_2 \lceil x_2 \rfloor^{\frac{1}{2}} + z \\ \dot{z} &= -k_3 \left\lceil x_1 + k_4 \lceil x_2 \rfloor^{\frac{3}{2}} \right\rfloor^{0} \end{aligned}$$

Closed Loop System:

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -k_1 \lceil x_1 \rfloor^{\frac{1}{3}} - k_2 \lceil x_2 \rfloor^{\frac{1}{2}} + z + \rho(t)\ , \\ \dot{z} &= -k_3 \left\lceil x_1 + k_4 \lceil x_2 \rfloor^{\frac{3}{2}} \right\rfloor^{0} \end{aligned}$$

Homogeneous Integral + State Feedback Controller

Discontinuous Integral Controller ($k_1, k_2, k_3 > 0$, $k_4 \in \mathbb{R}$, $L > 0$)

$$
\begin{aligned}
u &= -k_1 L^{\frac{2}{3}} \lceil x_1 \rfloor^{\frac{1}{3}} - k_2 L^{\frac{1}{2}} \lceil x_2 \rfloor^{\frac{1}{2}} + z \\
\dot{z} &= -k_3 L \left\lceil x_1 + k_4 L^{-\frac{3}{2}} \lceil x_2 \rfloor^{\frac{3}{2}} \right\rfloor^{0}
\end{aligned}
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$L > 0$ scaling gain:
If $\rho(t) = 0$: Stability for $L = 1 \Rightarrow$ Stability for any $L > 0$.

Remarks

- In contrast to the continuous Integral Controller:
  - It tracks exactly, in finite time and robustly
  - arbitrary references with bounded $\ddot{r}(t)$
  - despite arbitrary (time) Lipschitz perturbations/uncertainties, i.e. $\|\dot{\rho}(t)\| \leq \Delta$, Δ constant
  - without an Internal Model.
- For implementation: $r(t)$ and $\dot{r}(t)$ are required but not $\ddot{r}(t)$.
- Define $x_3 = z + \rho$. After convergence $\Rightarrow x(t) = 0 \Rightarrow$ $z(t) = -\rho(t)$: Integral action estimates the perturbation!
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Related controllers

- A similar algorithm is the "Continuous Twisting Algorithm".The proof is based on a Generalized Forms technique.

$$
\begin{aligned}
u &= -k_1 \lceil x_1 \rfloor^{\frac{1}{3}} - k_2 \lceil x_2 \rfloor^{\frac{1}{2}} + z \\
\dot{z} &= -k_3 \lceil x_1 \rfloor^{0} - k_4 \lceil x_2 \rfloor^{0}
\end{aligned}
$$

- The "High-Order Super Twisting"

$$
\begin{aligned}
u &= -k_1 \left\lceil x_2 + k_2 \lceil x_1 \rfloor^{\frac{2}{3}} \right\rfloor^{\frac{1}{2}} + z \\
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Outline

Homogeneous and smooth Lyapunov Function

Homogeneous and smooth Lyapunov Function $(L = 1)$

$$V_1\left(x_1,\, x_2;\, x_3\right) = k_1\left(\frac{3}{4}\left|x_1\right|^{\frac{4}{3}} - x_1\frac{x_3}{k_1} + \frac{1}{4}\left|\frac{x_3}{k_1}\right|^4\right) + \frac{1}{2}x_2^2\,,$$

$$V\left(x_1,\, x_2,\, x_3\right) \;=\; \gamma V_1^{\frac{5}{4}}\left(x_1,\, x_2;\, x_3\right) + \left(x_1 - \left(\frac{x_3}{k_1}\right)^3\right)x_2 + \frac{\mu}{5}\left|x_3\right|^5\,.$$

Its derivative is given by

$$\dot{V}\left(x_1,\, x_2,\, x_3\right) \;\in\; -W_1\left(x_1,\, x_2;\, x_3\right) - k_3W_2\left(x_1,\, x_2;\, x_3\right)$$

- The Lyapunov function fulfills following differential inequality
$$\dot{V}(x) \leq -\kappa V^{\frac{4}{5}}(x)\,,$$
for some $\kappa > 0$ depending on the gains and Δ.
- It implies robust finite time stability.
- Convergence time estimation:
$$T(x_0) \leq \frac{5}{\kappa} V^{\frac{1}{5}}(x_0)\,.$$

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Outline

Caveat

Alternative *Integral + state feedback* controllers:

- Linear Integral + state feedback controller (Homogeneous)

$$\begin{aligned} u &= -k_1 x_1 - k_2 x_2 + x_3 \\ \dot{x}_3 &= -k_3 x_1 \end{aligned}$$

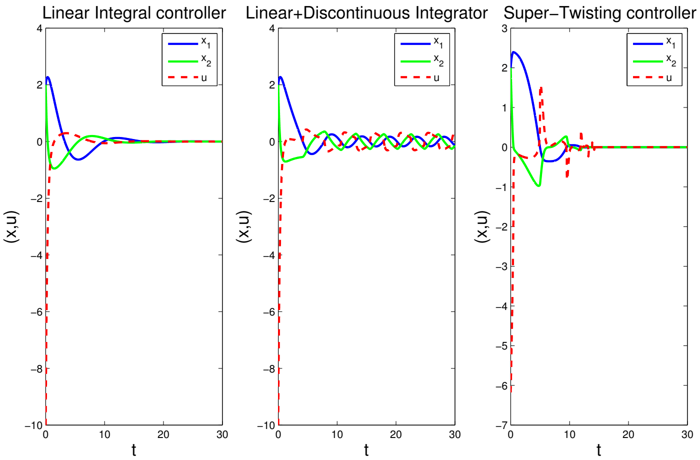
- Linear state feedback + Discontinuous Integral controller (Not Homogeneous)

$$\begin{aligned} u &= -k_1 x_1 - k_2 x_2 + x_3 \\ \dot{x}_3 &= -k_3 \mathrm{sign}(x_1) \end{aligned}$$

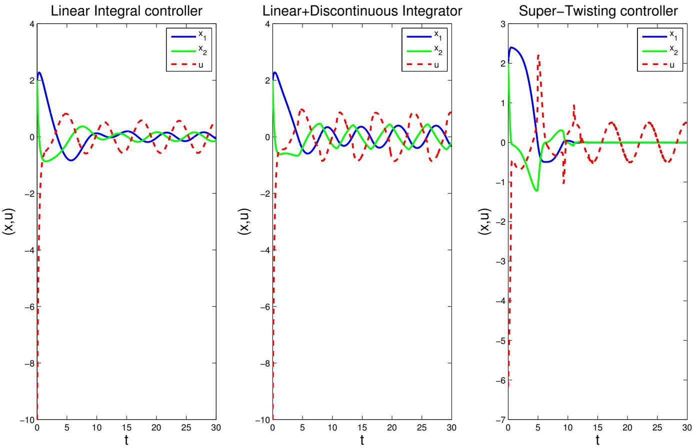
- Discontinuous I-Controller (Extended Super-Twisting) (Homogeneous)

$$\begin{aligned} u &= -k_1 |x_1|^{\frac{1}{3}} \mathrm{sign}(x_1) - k_2 |x_2|^{\frac{1}{2}} \mathrm{sign}(x_2) + x_3 \\ \dot{x}_3 &= -k_3 \mathrm{sign}(x_1) \end{aligned}$$

Controller without perturbation



Controller with perturbation



Remarks

- Linear stabilizes exponentially and is not insensitive to perturbation
- Linear + Discontinuous Integrator causes oscillations (Harmonic Balance). This is structural and for any $n > 2$. Eliminated by Homogeneity.
- Extended ST: Convergence in finite time and insensitive to perturbations.

Overview

Homogeneous Output Feedback Controller

Homogeneous State Feedback Controller + Homogeneous Observer

$$
\begin{aligned}
\dot{\hat{x}}_1 &= -l_1 \lceil \hat{x}_1 - x_1 \rfloor^{\frac{2}{3}} + \hat{x}_2 \\
\dot{\hat{x}}_2 &= -l_2 \lceil \hat{x}_1 - x_1 \rfloor^{\frac{1}{3}} - k_1 \lceil x_1 \rfloor^{\frac{1}{3}} - k_2 \lceil \hat{x}_2 \rfloor^{\frac{1}{2}} \\
u &= -k_1 \lceil x_1 \rfloor^{\frac{1}{3}} - k_2 \lceil \hat{x}_2 \rfloor^{\frac{1}{2}} .
\end{aligned}
$$

Homogeneous Integral + Output Feedback Controller

$$
\begin{aligned}
\dot{\hat{x}}_1 &= -l_1 \lceil \hat{x}_1 - x_1 \rfloor^{\frac{2}{3}} + \hat{x}_2 \\
\dot{\hat{x}}_2 &= -l_2 \lceil \hat{x}_1 - x_1 \rfloor^{\frac{1}{3}} - k_1 \lceil x_1 \rfloor^{\frac{1}{3}} - k_2 \lceil \hat{x}_2 \rfloor^{\frac{1}{2}} \\
u &= -k_1 \lceil x_1 \rfloor^{\frac{1}{3}} - k_2 \lceil \hat{x}_2 \rfloor^{\frac{1}{2}} + z \\
\dot{z} &= -k_3 \left\lceil x_1 + k_4 \lceil \hat{x}_2 \rfloor^{\frac{3}{2}} \right\rfloor^{0} ,
\end{aligned}
$$

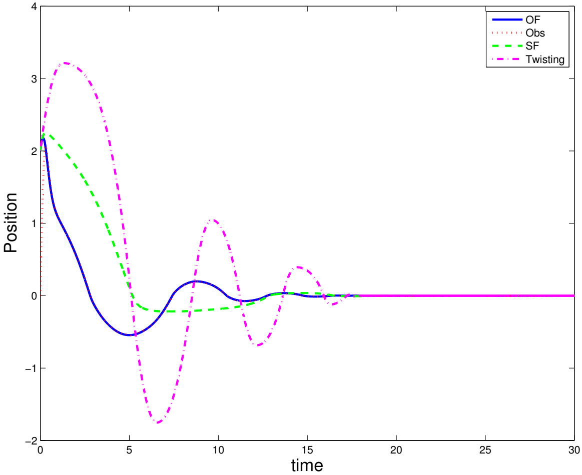
Simulations

We have implemented three controllers:

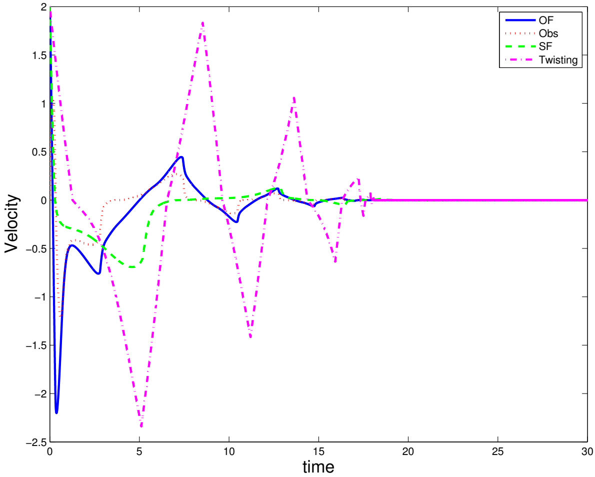
- A State Feedback (SF) controller with discontinuous integral term, with gains $k_1 = 2$, $k_2 = 5$, $k_3 = 0.5$, $k_4 = 0$, and initial value of the integrator $z(0) = 0$.
- An Output Feedback (OF) controller with discontinuous integral term, with controller gains $k_1 = 2\lambda^{\frac{2}{3}}$, $k_2 = 5\lambda^{\frac{1}{2}}$, $k_3 = 0.5\lambda$, $k_4 = 0$, $\lambda = 3$, observer gains $l_1 = 2L$, $l_2 = 1.1L^2$, $L = 4$, observer initial conditions $\hat{x}_1(0) = 0$, $\hat{x}_2(0) = 0$, and initial value of the integrator $z(0) = 0$.
- A Twisting controller, given by $u = -k_1 \lceil x_1 \rfloor^0 - k_2 \lceil x_2 \rfloor^0$, with gains $k_1 = 1.2$, $k_2 = 0.6$.

Perturbation $\rho(t) = 0.4 \sin(t)$

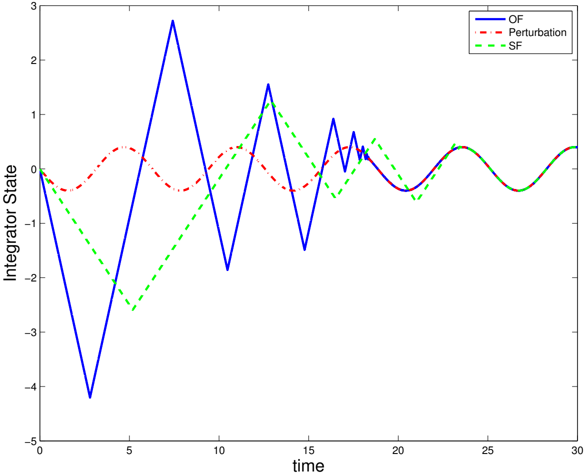
Simulations



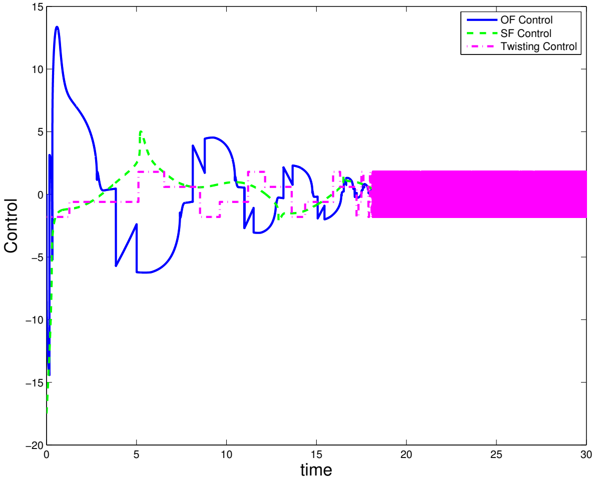
Simulations



Simulations



Simulations



Overview

Conclusions

- The Discontinuous Integral Controller
  - tracks exactly, in finite time and robustly
  - arbitrary references with bounded $\ddot{r}(t)$
  - despite arbitrary (time) Lipschitz perturbations/uncertainties, i.e. $\|\dot{\rho}(t)\| \leq \Delta$, Δ constant
  - without an Internal Model.
- Separate design of State Feedback and Observer;
- Neither continuous Observer nor continuous State Feedback Controller are insensitive to perturbations;
- Insensitivity against perturbations is achieved by discontinuous Integral Control;
- For implementation: $r(t)$ is required but not $\dot{r}(t)$ and $\ddot{r}(t)$.
- Design is Lyapunov-Based.
- Generalization to arbitrary order possible.

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Overview

Conclusions

1. Development of a Lyapunov based approach to HOSM and homogeneous control is an important task.
2. We require *constructive* methods to efficiently design controllers and observers for this class of systems.
3. We have provided some possible approaches. Each has its strengths and its weaknesses.
4. Still a lot of work has to be done
5. Other interesting approaches: Implicite Lyapunov Functions (ILF) by Lille Group!

Open Problems

1. Gain (and Structure) Design for Performance.
2. Performance comparison of HOSM Controllers with other controllers (e.g. FOSM).
3. Is there a family of LF for HOSM providing necessary and sufficient stability conditions ? Towards a more systematic Lyapunov Design.
4. (Truly) multivariable HOSM controllers and Observers. Some results for ST from Ch. Edwards,...
5. Adaptive Algorithms. Important results from Y. Shtessel, F. Plestan, ...
6. Parameter estimation...
7. Implementation of HOSMs, Discretization methods,...

Thank you! Gracias!

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