Sliding Mode Control à la Lyapunov

Jaime A. Moreno

Universidad Nacional Autónoma de México
Eléctrica y Computación, Instituto de Ingeniería,
04510 México D.F., Mexico, JMorenoP@ii.unam.mx

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Part I

HOSM Control and Homogeneity
Outline

1 Preliminaries

2 HOSM Control Problem
   - $\rho = 1$, First Order Sliding Mode (FOSM) Control Problem
   - $\rho = 2$, Second Order Sliding Mode (SOSM) Control

3 Homogeneity
   - Classical Homogeneity
   - Weighted Homogeneity
   - Weighted Homogeneity for systems with inputs (perturbations)
   - Weighted Homogeneity and Precision under perturbations
   - Homogeneous Approximation/Domination
   - Example: ”Danger” of Non Homogeneous Controllers

4 Homogeneous Design of HOSM (Levant 2005)

5 Plaidoyer for a Lyapunov-Based Framework for HOSM
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Consider a dynamical system

\[ \dot{x} = f(x). \]

We know that

- If \( f(x) \) discontinuous according to Filippov we obtain a DI
  \[ \dot{x} \in F(x), \quad F(x) \subset \mathbb{R}^n. \]

- If \( f(x) \) is uncertain, i.e. \( \|f(x)\| \leq f^+ \) we can write
  \[ \dot{x} \in [-f^+, f^+] \Rightarrow \dot{x} \in F(x). \]

- In case of discontinuity or/and uncertainty we obtain a Differential Inclusion from a Differential Equation
Filippov Differential Inclusions (DI)

\[ \dot{x} \in F(x) \]

is Filippov DI if \( \forall x \in \mathbb{R}^n \), the set-valued function \( F(x) \subset \mathbb{R}^n \) is

- not empty;
- compact;
- convex;
- upper-semicontinuous, i.e.

\[
\lim_{y \to x} \sup \{ \text{dist}(z, F(x)) | z \in F(y) \} = 0
\]

where

\[
\text{dist}(x, A) = \inf \{ ||x - a|| | a \in A \}.
\]

- A solution \( x(t) \) is an absolutely continuous function satisfying the DI almost everywhere.
- Filippov DI have the usual properties, except for uniqueness.
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Higher Order Sliding Mode (HOSM)

Consider a Filippov DI \( \dot{x} \in F(x) \), with a smooth output function \( \sigma = \sigma(x) \). If

1. The total time derivatives \( \sigma, \dot{\sigma}, \ldots, \sigma^{(r-1)} \) are continuous functions of \( x \)

2. The set

\[
\sigma = \dot{\sigma} = \ldots = \sigma^{(r-1)} = 0 \tag{1}
\]

is a nonempty integral set (i.e., consists of Filippov trajectories)

3. The Filippov set of admissible velocities at the \( r \)-sliding points (1) contains more than one vector

the motion on the set (1) is said to exist in an \( r \)-sliding (\( r \)th-order sliding) mode. The set (1) is called the \( r \)-sliding set.

The nonautonomous case is reduced to the one considered above by introducing the fictitious equation \( \dot{t} = 1 \).
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HOSM Control

SISO smooth, uncertain system

\[ \dot{z} = f(t, z) + g(t, z) u, \quad \sigma = h(t, z), \]

- \( z \in \mathbb{R}^n, u \in \mathbb{R}, \sigma \in \mathbb{R} \): sliding variable.
- \( f(t, z) \) and \( g(t, z) \) and \( n \) uncertain.
- **Control objective:** to reach and keep \( \sigma \equiv 0 \) in finite time.
- Relative Degree \( \rho \) w.r.t. \( \sigma \) is well defined, known and constant.
- Reduced (Zero) Dynamics asymptotically stable (by appropriate selection of \( \sigma \)).
The basic DI

Defining \( x = (x_1, \ldots, x_\rho)^T = (\sigma, \dot{\sigma}, \ldots, \sigma^{(\rho-1)})^T \), \( \sigma^i = \frac{d^i}{dt^i} h(z, t) \)

The regular form

\[
\sum_T: \begin{cases} 
\dot{x}_i = x_{i+1}, & i = 1, \ldots, \rho - 1, \\
\dot{x}_\rho = w(t, z) + b(t, z) u, & x_0 = x(0), \\
\dot{\zeta} = \phi(\zeta, x) & \zeta_0 = \zeta(0),
\end{cases}
\]

\( 0 < K_m \leq b(t, z) \leq K_M, |w(t, z)| \leq C. \)

Reduced Dynamics Asymptotically stable:

\( \dot{\zeta} = \phi(\zeta, 0) \zeta_0 = \zeta(0), \)

The basic Differential Inclusion (DI)

\[
\sum_{DI}: \begin{cases} 
\dot{x}_i = x_{i+1}, & i = 1, \ldots, \rho - 1, \\
\dot{x}_\rho \in [-C, C] + [K_m, K_M] u.
\end{cases}
\]
The Basic Problems

Bounded memoryless feedback controller

\[ u = \vartheta_\rho(x_1, x_2, \cdots, x_\rho), \]

- Render \( x_1 = x_2 = \cdots = x_\rho = 0 \) finite-time stable.
- Motion on the set \( x = 0 \) is \( \rho \text{th-order sliding mode} \).
- \( \vartheta_\rho \) necessarily discontinuous at \( x = 0 \) for robustness \([-C, C]\).

Problem 1
How to design an appropriate control law \( \vartheta_r \)?

Problem 2
How to estimate in finite time the required derivatives
\[ x = (x_1, \ldots, x_\rho)^T = (\sigma, \dot{\sigma}, \ldots, \sigma^{(\rho-1)})^T? \]
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$\rho = 1$, First Order Sliding Mode (FOSM) Control

- Introduced in the mid 60’s: Fantastic results, mature theory, lots of applications,... Utkin, Emelianov, ..... Edwards, Spurgeon, Sira-Ramirez, Loukianov, Zinober, ....
- $\dot{\sigma} \in [-C, C] + [K_m, K_M]u$.
- $u = \vartheta_1 = -k \text{sign}(\sigma), \ k > \frac{C}{K_m}$.
- Robust (=insensitive) control, simple realization, finite time convergence to the sliding manifold,...
- Lyapunov design using $V(\sigma) = \frac{1}{2}\sigma^2$.
- No derivative estimation required.
- Mature theory including Multivariable case, adaptation, design of sliding surfaces, ...
Sliding modes

$s(x) = 0$

grad $s$

Convex Hull

Yu.I. Neimark, Note on A. Filippov's paper, 1st IFAC Congress.

$dx/dt = Ax + bu + dv, u = -\text{sign}(s), v = -\text{sign}(s)$

Nonuniqueness!
Continuous vs. Discontinuous Control: A first order plant

Consider a plant

\[ \dot{\sigma} = \alpha + u, \quad \sigma(0) = 1 \]

where \( \alpha \in (-1, 1) \) is a perturbation.

Continuous (linear) Control

\[ \dot{\sigma} = \alpha - k\sigma, \quad k > 0, \quad \sigma(0) = 1 \]

Comments:

- RHS of DE continuous (linear).
- If \( \alpha = 0 \) exponential (asymptotic) convergence to \( \sigma = 0 \).
- If \( \alpha \neq 0 \) practical convergence.
Discontinuous Control

\[ \dot{\sigma} = \alpha - \text{sign}(\sigma), \quad \sigma(0) = 1 \]

with \( \alpha \in (-1, 1) \).

- \( \sigma > 0 \implies \dot{\sigma} = < 0 \)
- \( \sigma < 0 \implies \dot{\sigma} = > 0 \)

\( y \sigma(t) \equiv 0, \forall t \geq T. \)

Comments:

- \( 0 = \alpha - \text{sign}(0) \)?
- RHS of DE is discontinuous.
- After arriving at \( \sigma = 0 \), sliding on \( \sigma \equiv 0 \).
- Finite-Time convergence.
- Differential Inclusion.
Notice the **chattering = infinite switching** of the control variable!
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\(\rho = 2, \text{ Second Order SM (SOSM) Control}\)

- Introduced in the mid 80’s: Wonderful results, mature geometric theory, lots of applications,... Levant, Fridman, Bartolini, Ferrara, Shtessel, Usai, Feng, Man, Yu, Furuta, Spurgeon, Orlov, Perruquetti, Barbot, Floquet, Defoort, ....

- Some controllers (see Fridman 2011):
  - Twisting Controller (Emelyanov, Korovin, Levant 1986).
    \[\vartheta_2(x) = -k_1\text{sign}(x_1) - k_2\text{sign}(x_2).\]
  - The Sub-Optimal Algorithm (Bartolini, Ferrara, Usai 1997).
  - Terminal Sliding Mode Control (Man, Paplinski, Wu, Yu (1994, 1997..)).

\[
\sum_{DI} : \left\{ \begin{array}{l}
\dot{x}_1 = x_2,
\dot{x}_2 \in [-C, C] + [K_m, K_M]u.
\end{array} \right.
\]
A second order plant

\[
\begin{align*}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= \phi(x_1, x_2) + u \\
y &= x_1
\end{align*}
\]

\(\phi\): Perturbation/uncertainty.

**Question**: Can we just feedback the output (as for FO case)?

Two alternative *output* controllers:

- **Continuous (linear) output controller** (*Homogeneous Time Invariant* (HTI))
  \[
  \begin{align*}
  \dot{x}_1 &= x_2 \\
  \dot{x}_2 &= -k_1 y
  \end{align*}
  \]

- **Discontinuous output controller** (*HTI*)
  \[
  \begin{align*}
  \dot{x}_1 &= x_2 \\
  \dot{x}_2 &= -k_1 \text{sign}(y)
  \end{align*}
  \]
Both Oscillate!

It is \textbf{impossible} to stabilize a double (or triple etc) integrator by static output feedback!
State feedback

State feedback controllers:

- Continuous (linear) state feedback controller (HTI)

\[
\begin{align*}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= -k_1 x_1 - k_2 x_2
\end{align*}
\]

- Exponential Convergence
- Robust, but Sensitive to perturbations: Practical stability.

- Continuous HTI state feedback controller

\[
\begin{align*}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= -k_1 \left| x_1 \right|^{\frac{1}{3}} - k_2 \left| x_2 \right|^{\frac{1}{2}}
\end{align*}
\]

\[
\left| \cdot \right|^\rho = \left| \cdot \right|^\rho \text{sign}(\cdot)
\]

- Finite Time Convergence
- Robust, but Sensitive to perturbations: Practical stability.
First Order Sliding Mode Controller

\[
\begin{align*}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= -k_2 \text{sign}(x_2 + k_1 x_1)
\end{align*}
\]

Rewrite as a first order system with a stable (first order) zero dynamics: with \( \sigma = x_2 + k_1 x_1 \) sliding variable

\[
\begin{align*}
\dot{x}_1 &= -k_1 x_1 + \sigma \\
\dot{\sigma} &= -k_2 \text{sign}(\sigma)
\end{align*}
\]

\[\sigma = x_2 + c x_1 = 0\]
- Linear controller stabilize exponentially and is not insensitive to perturbation
- SM control also stabilizes exponentially but is insensitive to perturbation!
The Twisting Controller

A discontinuous HTI controller able to obtain finite-time convergence and insensitivity to perturbations:

\[
\begin{align*}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= -k_1 \text{sign}(x_1) - k_2 \text{sign}(x_2)
\end{align*}
\]
$\rho = 2$, SOSM Control....

- Robust (=insensitive) control, finite time convergence to the sliding manifold,...
- No Lyapunov design or analysis.
- Analysis and Design is very geometric: Beautiful but difficult to extend to $\rho > 2$.
- **Solution**: Homogeneity!
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Classical Homogeneity

Classical Homogeneity for functions. (Euler, Zubov, Hahn,...)

Let $n, m$ be positive integers. A mapping $f : \mathbb{R}^n \to \mathbb{R}^m$ is homogeneous with degree $\delta \in \mathbb{R}$ iff $\forall \lambda > 0$:

$$f(\lambda x) = \lambda^\delta f(x).$$

Some examples:

- Linear function: Let $A \in \mathbb{R}^{m \times n}$ then $f(x) = Ax$ is homogeneous of degree $\delta = 1$, since

$$f(\lambda x) = A(\lambda x) = \lambda (Ax) = \lambda f(x).$$

- $f(x_1, x_2) = \frac{x_1^3 + x_2^3}{x_1^2 + x_2^2}$ is continuous and homogeneous of degree $\delta = 1$ (but not linear!), since

$$f(\lambda x_1, \lambda x_2) = \frac{(\lambda x_1)^3 + (\lambda x_2)^3}{(\lambda x_1)^2 + (\lambda x_2)^2} = \frac{\lambda^3 (x_1^3 + x_2^3)}{\lambda^2 (x_1^2 + x_2^2)} = \lambda f(x_1, x_2).$$
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\]
\( f(x_1, x_2) = \begin{cases} \frac{\lfloor x_1 \rfloor^{\frac{1}{2}} + \lfloor x_2 \rfloor^{\frac{1}{2}}}{x_1 + x_2} & \text{if } x_1 + x_2 \neq 0 \\ 0 & \text{otherwise} \end{cases} \)

is discontinuous and homogeneous of degree \( \delta = -\frac{1}{2} \), since

\[ f(\lambda x_1, \lambda x_2) = \frac{\lfloor \lambda x_1 \rfloor^{\frac{1}{2}} + \lfloor \lambda x_2 \rfloor^{\frac{1}{2}}}{\lambda x_1 + \lambda x_2} = \lambda^{-\frac{1}{2}} f(\lambda x_1, \lambda x_2) \]

- **Quadratic Form**: if \( x \in \mathbb{R}^n \) and \( P \in \mathbb{R}^{n \times n} \), \( q(x) = x^T P x \) is homogeneous of degree \( \delta = 2 \), since

\[ q(\lambda x) = (\lambda x)^T P(\lambda x) = \lambda^2 x^T P x = \lambda^2 q(x) . \]

- **Classical Form = homogeneous polynomial**: if \( x \in \mathbb{R}^n \), e.g.

\[ p(x) = \alpha_1 x_1 x_2 x_3 + \alpha_2 x_1 x_3 x_5 + \alpha_3 x_1^2 x_5 + \alpha_4 x_2^3 + \cdots \]

is homogeneous of degree \( \delta = 3 \).
\[ f(x_1, x_2) = \begin{cases} 
\frac{[x_1]^{\frac{1}{2}} + [x_2]^{\frac{1}{2}}}{x_1 + x_2} & \text{if } x_1 + x_2 \neq 0 \\
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**Classical Form = homogeneous polynomial:** if \( x \in \mathbb{R}^n \), e.g.

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is homogeneous of degree \( \delta = 3 \).
Classical Homogeneity for vector fields. (Zubov, Hahn,...)

Let $n$ be a positive integer. A vector field $f : \mathbb{R}^n \to \mathbb{R}^n$ is homogeneous with degree $\delta \in \mathbb{R}$ iff $\forall \lambda > 0$:

$$f(\lambda x) = \lambda^\delta f(x).$$

Associated with the vector field $f(x)$ is the Differential Equation $\dot{x} = f(x)$, and it has a flow (solution) $\varphi(t, x)$.

Homogeneity of vector field $\Rightarrow$ Homogeneity of Flow

If $\forall \lambda > 0$:

$$f(\lambda x) = \lambda^\delta f(x) \Rightarrow \varphi(t, \lambda x) = \lambda \varphi(\lambda^{\delta-1}t, x)$$
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Homogeneity of vector field $\Rightarrow$ Homogeneity of Flow

If $\forall \lambda > 0$:

$$f(\lambda x) = \lambda^\delta f(x) \Rightarrow \varphi(t, \lambda x) = \lambda \varphi(\lambda^{\delta-1} t, x)$$
Some examples:

- Linear system: Let $A \in \mathbb{R}^{n \times n}$ then $\dot{x} = f(x) = Ax$ is homogeneous of degree $\delta = 1$ and the flow is $\varphi(t, x) = e^{At}x$

$$
\varphi(t, \lambda x) = e^{At}(\lambda x) = \lambda e^{At}x = \lambda \varphi(\lambda^0 t, x).
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- If $x$ is scalar. System $\dot{x} = -\text{sign}(x)$ is homogeneous with degree $\delta = 0$, since

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\varphi(t, x) = \begin{cases}
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Outline

1 Preliminaries

2 HOSM Control Problem
   - $\rho = 1$, First Order Sliding Mode (FOSM) Control Problem
   - $\rho = 2$, Second Order Sliding Mode (SOSM) Control

3 Homogeneity
   - Classical Homogeneity
   - Weighted Homogeneity
     - Weighted Homogeneity for systems with inputs (perturbations)
     - Weighted Homogeneity and Precision under perturbations
   - Homogeneous Approximation/Domination
   - Example: ”Danger” of Non Homogeneous Controllers

4 Homogeneous Design of HOSM (Levant 2005)

5 Plaidoyer for a Lyapunov-Based Framework for HOSM
Weighted Homogeneity or
(quasi-homogeneity) (Zubov, Hermes)

- A generalized **weight** is a vector \( \mathbf{r} = (r_1, \cdots, r_n) \), with \( r_i > 0 \).
- A **dilation** is the action of the group \( \mathbb{R}_+ \setminus \{0\} \) on \( \mathbb{R}^n \) given by

\[
\Lambda_{\mathbf{r}} : \mathbb{R}_+ \setminus \{0\} \times \mathbb{R}^n \rightarrow \mathbb{R}^n
\]

\[
(\lambda, x) \rightarrow \text{diag}\{\lambda^{r_i}\} x
\]

we will denote this for simplicity as \( \Lambda_{\mathbf{r}} x \triangleq \Lambda_{\mathbf{r}}(\lambda, x), \lambda > 0 \).

Weighted Homogeneity for functions. (Zubov, Hermes...)

Let \( n, m \) be positive integers. A mapping \( f : \mathbb{R}^n \rightarrow \mathbb{R}^m \) is \( \mathbf{r} \)-homogeneous with **degree** \( \delta \in \mathbb{R} \) iff \( \forall \lambda > 0 : \)

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Some remarks

- Function $f(x_1, x_2) = x_1 + x_2^2$ is not homogeneous, but it is $(2, 1)$-homogeneous of degree $\delta = 2$ since,

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- Classical Homogeneity = Weighted Homogeneity with $r = (r_1, \cdots, r_n) = (1, \cdots, 1)$.

- Values on the unit sphere define an $r$-Homogeneous function.

- If $f(x)$ is $r$-homogeneous of degree $\delta$ then it is $(\alpha r)$-homogeneous of degree $(\alpha \delta)$ for any $\alpha > 0$.

- Euler's Theorem: Let $V : \mathbb{R}^n \to \mathbb{R}$ be differentiable. $V$ is $r$-homogeneous of degree $\delta$ if and only if

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Lyapunov SMC      Jaime A. Moreno      UNAM 36
Weighted Homogeneity for (set-valued) vector fields. (Zubov, Hermes, Levant, ...)

Let $n$ be a positive integer. A vector field $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$ (a set-valued vector field $f : \mathbb{R}^n \Rightarrow \mathbb{R}^n$) is $r$-homogeneous with degree $\delta \in \mathbb{R}$ iff $\forall \lambda > 0$:

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Associated with the (set-valued) vector field $f(x)$ is the Differential Equation $\dot{x} = f(x)$ (DI $\dot{x} \in f(x)$), and it has a flow (solution) $\varphi(t, x)$.

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If $\forall \lambda > 0$:

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Some examples:

- **Super-Twisting (ST) Algorithm:**
  \[
  \dot{x}_1 = -k_1 \left[ x_1 \right]^{\frac{1}{2}} + x_2 \\
  \dot{x}_2 \in -k_2 \left[ x_1 \right]^0 + [-1, 1].
  \]

  is $(2, 1)$-homogeneous of degree $-1$, since

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  -k_1 \left[ \lambda^2 x_1 \right]^{\frac{1}{2}} + \lambda x_2 = \lambda^{2-1}(-k_1 \left[ x_1 \right]^{\frac{1}{2}} + x_2) \\
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Two important examples:

- Levant’s arbitrary order differentiator:

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\dot{x}_i = -k_i \left[ x_1 - f(t) \right]^{\frac{n-i}{n}} + x_{i+1} \\
\vdots \\
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- Homogeneous Controller of a chain of integrators:

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Dynamic interpretation of r-homogeneity

- System $\dot{x} = f(x)$ is $r$-homogeneous of degree $\delta$, i.e. $f(\Lambda_r x) = \lambda^\delta \Lambda_r f(x)$.
- State Transformation \( z = \Lambda_r x \)
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  \dot{z} = \Lambda_r \dot{x} = \Lambda_r f(x) = \lambda^{-\delta} f(\Lambda_r x)
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  and therefore
  \[
  \frac{dz}{d(\lambda^\delta t)} = f(z)
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Properties of Homogeneous Systems

Zubov, Hahn, Hermes, Kawski, Rosier, Aeyels, Sepulchre, Grüne, Praly, Perruquetti, Efimov, Polyakov,..... Levant, Orlov, Bernuau et al. 2013

- If $x = 0$ Locally Attractive (LA) $\Leftrightarrow$ Globally Asymptotically Stable (GAS)
- local contraction $\Rightarrow$ global contraction $\Rightarrow$ global asymptotic stability
- If $x = 0$ GAS and $\delta < 0$ $\Leftrightarrow$ $x = 0$ Finite Time Stable
- If $x = 0$ GAS and $\delta = 0$ $\Leftrightarrow$ $x = 0$ Exponentially Stable (e.g. LTI systems)
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- If $x = 0$ GAS and $\delta > 0$ $\iff$ $x = 0$ Asymptotically Stable

- If $x = 0$ GAS $\iff$ It exists a Homogeneous Lyapunov Function
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2 HOSM Control Problem
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   - $\rho = 2$, Second Order Sliding Mode (SOSM) Control

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   - Classical Homogeneity
   - Weighted Homogeneity
     - Weighted Homogeneity for systems with inputs (perturbations)
     - Weighted Homogeneity and Precision under perturbations
   - Homogeneous Approximation/Domination
   - Example: ”Danger” of Non Homogeneous Controllers

4 Homogeneous Design of HOSM (Levant 2005)

5 Plaidoyer for a Lyapunov-Based Framework for HOSM
Weighted Homogeneity with inputs

Consider a System with inputs \( u \in \mathbb{R}^m \)

\[
\dot{x} = f(x, u).
\]

- State and input weight vectors \( \mathbf{r} = (r_1, \cdots, r_n), r_i > 0, \)
  \( \rho = (\rho_1, \cdots, \rho_m), \rho_i > 0 \)
- State and input dilations \( \Lambda_{\mathbf{r}} \) and \( \Lambda_{\rho}. \)

Weighted Homogeneity for (set-valued) vector fields with inputs.

A vector field \( f : \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}^n \) (a set-valued vector field \( f : \mathbb{R}^n \times \mathbb{R}^m \Rightarrow \mathbb{R}^n \)) is \((\mathbf{r}, \rho)\)-homogeneous with degree \( \delta \in \mathbb{R} \) iff \( \forall \lambda > 0: \)

\[
f(\Lambda_{\mathbf{r}} x, \Lambda_{\rho} u) = \lambda^\delta \Lambda_{\mathbf{r}} f(x, u).
\]
Weighted Homogeneity with inputs

Consider a System with inputs $u \in \mathbb{R}^m$

$$\dot{x} = f(x, u).$$

- State and input weight vectors $\mathbf{r} = (r_1, \cdots, r_n), r_i > 0,$
  $\mathbf{r} = (\rho_1, \cdots, \rho_m), \rho_i > 0$
- State and input dilations $\Lambda_{\mathbf{r}}$ and $\Lambda_{\mathbf{\rho}}.$

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$$f(\Lambda_{\mathbf{r}}x, \Lambda_{\mathbf{\rho}}u) = \lambda^\delta \Lambda_{\mathbf{r}}f(x, u).$$
Weighted Homogeneity with inputs

Consider a System with inputs $u \in \mathbb{R}^m$

$$\dot{x} = f(x, u).$$

- State and input weight vectors $r = (r_1, \cdots, r_n), r_i > 0$, $\rho = (\rho_1, \cdots, \rho_m), \rho_i > 0$
- State and input dilations $\Lambda_r$ and $\Lambda_\rho$.

Weighted Homogeneity for (set-valued) vector fields with inputs.

A vector field $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ (a set-valued vector field $f : \mathbb{R}^n \times \mathbb{R}^m \Rightarrow \mathbb{R}^n$) is $(r, \rho)$-homogeneous with degree $\delta \in \mathbb{R}$ iff

$$\forall \lambda > 0 :$$

$$f(\Lambda_r x, \Lambda_\rho u) = \lambda^\delta \Lambda_r f(x, u).$$
Associated with the (set-valued) vector field \( f(x, u) \) is the Differential Equation \( \dot{x} = f(x, u) \) (\( \text{DI} \dot{x} \in f(x, u) \)), and it has a flow (solution) \( \varphi(t, x, u) \).

**Homogeneity of vector field \Rightarrow Homogeneity of Flow**

If \( f \) is homogeneous then \( \forall \lambda > 0 : \)

\[
\varphi(t, \Lambda_r x, \Lambda_\rho u(\lambda^\delta \cdot)) = \Lambda_r \varphi(\lambda^\delta t, x, u(\cdot))
\]
Dynamic interpretation of homogeneity

- System \( \dot{x} = f(x, u) \) is homogeneous of degree \( \delta \), i.e.
  \[ f(\Lambda_{\rho}x, \Lambda_{\rho}u) = \lambda^{\delta} \Lambda_{\rho}f(x, u). \]
- State and input Transformation \( z = \Lambda_{\rho}x, w = \Lambda_{\rho}u \)

\[ \dot{z} = \Lambda_{\rho}\dot{x} = \Lambda_{\rho}f(x, u) = \lambda^{-\delta} f(\Lambda_{\rho}x, \Lambda_{\rho}u) \]

and therefore

\[ \frac{dz}{d(\lambda^{\delta}t)} = f(z, w) \]

- System \( \dot{x} = f(x) \) is invariant under the transformation

\[ G_{\lambda} : (t, x, u) \mapsto (\lambda^{-\delta}t, \Lambda_{\rho}x, \Lambda_{\rho}u). \]

- Internal stability \( \Rightarrow \) external stability (iISS, ISS) [Bernuau et al. 2013]
Dynamic interpretation of homogeneity

- System $\dot{x} = f(x, u)$ is homogeneous of degree $\delta$, i.e.
  $f(\Lambda_r x, \Lambda_\rho u) = \lambda^\delta \Lambda_r f(x, u)$.

- State and input Transformation $z = \Lambda_r x$, $w = \Lambda_\rho u$
  $$\dot{z} = \Lambda_r \dot{x} = \Lambda_r f(x, u) = \lambda^{-\delta} f(\Lambda_r x, \Lambda_\rho u)$$
  and therefore
  $$\frac{dz}{d(\lambda^\delta t)} = f(z, w)$$

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  \[ f(\Lambda_\rho x, \Lambda_\rho u) = \lambda^\delta \Lambda_\rho f(x, u). \]

- State and input Transformation \( z = \Lambda_\rho x, w = \Lambda_\rho u \)
  \[ \dot{z} = \Lambda_\rho \dot{x} = \Lambda_\rho f(x, u) = \lambda^{-\delta} f(\Lambda_\rho x, \Lambda_\rho u) \]

  and therefore
  \[ \frac{dz}{d(\lambda^\delta t)} = f(z, w) \]

- System \( \dot{x} = f(x) \) is invariant under the transformation
  \[ G_\lambda : (t, x, u) \mapsto (\lambda^{-\delta} t, \Lambda_\rho x, \Lambda_\rho u). \]

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- System $\dot{x} = f(x, u)$ is homogeneous of degree $\delta$, i.e.
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- State and input Transformation $z = \Lambda_r x$, $w = \Lambda_\rho u$

  $$\dot{z} = \Lambda_r \dot{x} = \Lambda_r f(x, u) = \lambda^{-\delta} f(\Lambda_r x, \Lambda_\rho u)$$

  and therefore
  $$\frac{dz}{d(\lambda^\delta t)} = f(z, w)$$

- System $\dot{x} = f(x)$ is invariant under the transformation

  $$G_\lambda : (t, x, u) \mapsto (\lambda^{-\delta} t, \Lambda_r x, \Lambda_\rho u).$$

- Internal stability $\Rightarrow$ external stability (iISS, ISS) [Bernuau et al. 2013]
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4 Homogeneous Design of HOSM (Levant 2005)

5 Plaidoyer for a Lyapunov-Based Framework for HOSM
Consider a System with a scalar and constant input \( u \in \mathbb{R} \)

\[
\dot{x} = f(x, u).
\]

so that

\[
\varphi(t, \Lambda_t x, \lambda^\rho u) = \Lambda_t \varphi(\lambda^\delta t, x, u).
\]

Suppose that asymptotically or after a finite time for some \( u_0 \)

\[
\lim_{t \to \infty} |\varphi_i(t, x, u_0)| = |\varphi_\infty(u_0)| \leq a_i.
\]

Therefore (using \( \lambda = \left(\frac{u}{u_0}\right)^{\frac{1}{\rho}} \))

\[
|\varphi_\infty(u)| = |\varphi_\infty(\lambda^\rho u_0)| = \lambda^{r_i} |\varphi_\infty(u_0)| \leq \nu_i u^{\frac{r_i}{\rho}},
\]

with \( \nu_i = a_i u_0^{\frac{-r_i}{\rho}}. \)
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Homogeneous Approximation

- First Lyapunov’s Theorem: \( x = 0 \) is LAS for \( \dot{x} = f(x) \) if \( x = 0 \) is AS for linearized system \( \dot{x} = Ax \), where \( A = \frac{\partial f(0)}{\partial x} \).
- This is not true for Taylor Approximations. Example from Bacciotti & Rosier, 2005:
  \[
  \begin{align*}
  \dot{x}_1 &= -x_1^3 + x_2^3, \\
  \dot{x}_2 &= -x_1 + x_2^5.
  \end{align*}
  \]

  \( x = 0 \) is GAS for approximation of order 3 (black), but it is not AS with perturbation (red) of higher order 5!

- However, it is true for homogeneous approximations (Not unique!)
  \[
  \begin{align*}
  \dot{x}_1 &= -x_1^3 + x_2, \\
  \dot{x}_2 &= -x_1^5 + x_2^2.
  \end{align*}
  \]

  \( x = 0 \) is GAS for homogeneous approximation \( r_1 = 1, r_2 = 3, \delta = 2 \) (black), and it is still AS with perturbation (red), which is homogeneous of degree 3.
Homogeneous Domination

Typical control (similar for observation) problem:

\[ \dot{x}_1 = x_2 + f_1(x_1), \]
\[ \dot{x}_2 = x_3 + f_2(x_1, x_2), \]
\[ \vdots \]
\[ \dot{x}_n = u + f_n(x). \]

Homogeneous Approximation (Black):

\[ \dot{x}_1 = x_2, \]
\[ \vdots \]
\[ \dot{x}_n = u = \phi(x). \]

It is homogeneous of degree \( \delta = \{-1, 0, +1\} \) and weights \( r = (r_1 + \delta, r_1 + 2\delta, \cdots, r_1 + n\delta) \): NOT UNIQUE! \( \Rightarrow \) Fixed by selection of the control law \( \phi(x) \) \( \Rightarrow \) Domination of other terms \( f_i \).
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Alternative *Integral + state feedback* controllers for System

\[
\begin{align*}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= u + \rho(t),
\end{align*}
\]

- **Linear Integral + state feedback controller (Homogeneous)**
  
  \[
  u = -k_1 x_1 - k_2 x_2 + x_3 \\
  \dot{x}_3 = -k_3 x_1
  \]

- **Linear state feedback + Discontinuous Integral controller (Not Homogeneous)**
  
  \[
  u = -k_1 x_1 - k_2 x_2 + x_3 \\
  \dot{x}_3 = -k_3 \text{sign}(x_1)
  \]

- **Discontinuous I-Controller (Extended Super-Twisting) (Homogeneous)**
  
  \[
  u = -k_1 |x_1|^{\frac{1}{3}} \text{sign}(x_1) - k_2 |x_2|^{\frac{1}{2}} \text{sign}(x_2) + x_3 \\
  \dot{x}_3 = -k_3 \text{sign}(x_1)
  \]
Controller without perturbation

Linear Integral controller

Linear+Discontinuous Integrator

Super–Twisting controller

Lyapunov SMC  Jaime A. Moreno  UNAM
Controller with perturbation

![Graphs comparing different controllers](image)

- **Linear Integral controller**
- **Linear+Discontinuous Integrator**
- **Super–Twisting controller**

Lyapunov SMC  Jaime A. Moreno  UNAM
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\[
\sum_{DI} \cdot \left\{ \begin{array}{l}
\dot{x}_i = x_{i+1}, \ i = 1, \ldots, \rho - 1, \\
\dot{x}_\rho \in [-C, C] + [K_m, K_M]u.
\end{array} \right.
\]

\[
u_r(x_1, x_2, \ldots, x_\rho),
\]

- \( \nu_r \) homogeneous of degree 0 (discontinuous at \( x = 0 \)) with weights \( r_s = (\rho, \rho - 1, \ldots, 1) \)

\[
\nu_r (\epsilon^\rho x_1, \epsilon^{\rho-1} x_2, \ldots, \epsilon x_\rho) = \nu_r (x_1, x_2, \ldots, x_\rho) \ \forall \epsilon > 0
\]

- Local boundedness ⇒ global boundedness

- System \( \sum_{DI} \) is homogeneous of degree \(-1\) with weights \( r_s \)
  - Local contractive ⇔ Global, uniformly Finite-Time stability
  - Robustness of stability ⇒ Accuracy with respect to homogeneous perturbations \( |x_i| \leq \gamma_i \tau^{\rho+1-i} = O(\tau^{\rho+1-i}) \), \( \gamma_i \) constants.
Concrete Homogeneous HOSM Controllers

Notation: \( [x]^p = |x|^p \text{sign}(x) \).

**Nested Sliding Controllers (NSC)**

- \( u_{2L} = -k_2 \left[ x_2 + \beta_1 [x_1]^{1/2} \right]^0 \),
- \( u_{3L} = -k_3 \left[ x_3 + \beta_2 (|x_2|^3 + |x_1|^2)^{1/6} \left[ x_2 + \beta_1 [x_1]^{2/3} \right]^0 \right]^0 \),

**Quasi-Continuous Sliding Controllers (QCSC)**

- \( u_{2C} = -k_2 \frac{(x_2 + \beta_1 [x_1]^{1/2})}{|x_2| + \beta_1 |x_1|^{1/2}} \)
- \( u_{3C} = -k_3 \frac{x_3 + \beta_2 (|x_2| + \beta_1 |x_1|^{3/2})^{-1/2} (x_2 + \beta_1 [x_1]^{3/2})}{|x_3| + \beta_2 (|x_2| + \beta_1 |x_1|^{3/2})^{1/2}} \).
Scaling the gains: again homogeneity!

Example: 2nd Order Nested Controller

\[
\begin{align*}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= -k_2 \left[ x_2 + \beta_1 \left[ x_1 \right]^{\frac{1}{2}} \right]^0
\end{align*}
\]

Linear change of coordinates: \( z = \lambda^2 x, \ 0 < \lambda \in \mathbb{R} \)

\[
\begin{align*}
\dot{z}_1 &= \lambda^2 x_2 = z_2 \\
\dot{z}_2 &= -\lambda^2 k_2 \left[ \frac{z_2}{\lambda^2} + \beta_1 \left[ \frac{z_1}{\lambda^2} \right]^{\frac{1}{2}} \right]^0 \\
&= -\lambda^2 k_2 \left[ \frac{z_2}{\lambda} + \beta_1 \left[ z_1 \right]^{\frac{1}{2}} \right]^0
\end{align*}
\]

Preserves stability!
Example: 3rd Order Nested Controller

\[
\begin{align*}
\dot{x}_1 &= x_2, \quad \dot{x}_2 = x_3, \\
\dot{x}_3 &= -k_3 \left[ x_3 + \beta_2 (|x_2|^3 + |x_1|^2)^{\frac{1}{6}} \left[ x_2 + \beta_1 \left[ x_1^{\frac{2}{3}} \right] \right] \right]
\end{align*}
\]

\[z = \lambda^3 x, \quad 0 < \lambda \in \mathbb{R} \Rightarrow \text{Preserves stability!}\]

\[
\begin{align*}
\dot{z}_1 &= z_2, \quad \dot{z}_2 = z_3, \\
\dot{z}_3 &= -\lambda^3 k_3 \left[ \frac{z_3}{\lambda^3} + \beta_2 \left( \left| \frac{z_2}{\lambda^3} \right|^3 + \left| \frac{z_1}{\lambda^3} \right|^2 \right)^{\frac{1}{6}} \left[ \frac{z_2}{\lambda^3} + \beta_1 \left[ \frac{z_1}{\lambda^3}^{\frac{2}{3}} \right] \right] \right]
\end{align*}
\]

\[
\begin{align*}
\dot{z}_3 &= -\lambda^3 k_3 \left[ \frac{z_3}{\lambda^2} + \beta_2 \left( \left| \frac{z_2}{\lambda} \right|^3 + |z_1|^2 \right)^{\frac{1}{6}} \left[ \frac{z_2}{\lambda} + \beta_1 \left[ z_1^{\frac{2}{3}} \right] \right] \right]
\end{align*}
\]
Balance

Advantages:

- Beautiful and powerful theory: more qualitative than quantitative
- Simple: local = global, convergence = finite-time = robust
- scaling the gains (for nested controllers) for convergence acceleration

\[ u = \lambda^\rho \vartheta_r \left( x_1, \frac{x_2}{\lambda}, \ldots, \frac{x_\rho}{\lambda^{\rho-1}} \right), \quad \lambda > 1 \]
Balance....

**Limitations:**
- Beyond homogeneity unclear how to design $\vartheta_r$!
- It does not provide quantitative results, e.g.
  - Stabilizing Gains
  - Convergence Time estimation
  - Accuracy Gains
  - Performance quantities
- Behavior with respect to non homogeneous perturbations
- Behavior under interconnection
- Design for performance

Due to Limitations we require other methods, e.g. Lyapunov (but not exclusively)
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Motivation

Lyapunov based methods

- Standard methods in nonlinear control theory
- Design for robustness, optimality, performance, etc.
- Gain tuning methods
- Lyapunov Function for Internal Stability and Lyapunov-like Functions for External Stability (e.g. ISS, iISS,...)
- LF for design: Control Lyapunov Functions (CLF)
- Interconnection analysis is possible
- Robustness analysis: noise, uncertainties, perturbations

Objective:
A Lyapunov Based framework for HOSM

Belief: Combination of Homogeneity and LF ⇒ powerful tool!
Main Problem: Construction of LF

Existence of LF for $f(x)$ Filippov Differential Inclusion

- It exists a smooth p.d. $V(x)$ [Clarke et al. 1998]
- $f(x)$ homogeneous $\Rightarrow V(x)$ homogeneous [Nakamura et al. 2002, Bernuau et al. 2014]

There are basically two issues:

- What is the form or structure of the LF?
- How to decide if $V(x)$ and $W(x)$ are positive definite (p.d.)?

There are many works on these general topics. But there are few for HOSM algorithms and taking advantage of the homogeneity properties.
State of the art


Polyakov and Poznyak, 2009  Strong non smooth LF for Twisting and ST. Zubov’s Method.

Santiesteban et al., 2010  Strong LF for Twisting with linear terms. No method.


Sánchez and Moreno, 2012  Strong non smooth LF for Twisting, Terminal and a sign controller. Trajectory Integration method.
Some attempts (in our group)

- First steps: the quadratic form approach
- Young’s inequality and extensions
- Trajectory integration
- Reduction method
- Generalized Forms approach
- Homogenous Control Lyapunov Functions
- etc.

I will not talk about Lille’s Group Implicit Lyapunov Functions (ILF) approach (Polyakov, Efimov, Perruquetti,...)!
Part II

Lyapunov-Based Design of Higher-Order Sliding Mode (HOSM) Controllers
Outline

6 Control Lyapunov Functions and Families of HOSM CLFs

7 HOSM Controllers

8 HOSM Controllers: Some Examples

9 Gain Calculation

10 Simulation Example
Perturbed nonlinear system

\[ \dot{x} \in F(x) + g(x) \xi(x) u, \]

- \( x \in \mathbb{R}^n, u \in \mathbb{R} \)
- \( g(x) \) known vector field
- \( F(x) \) set-vector field, \( \xi \) multivalued \( \Rightarrow \) Uncertainties.

Assumptions

- \( 0 < K_m \leq \xi(x) \leq K_M \)
- \( F, g \) are \( r \)-homogeneous of degree \( l \).

\[ [5, 6, 13] \]
Overview

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(Homogeneous) CLF

\[ V(x) \in C^1, \text{ p.d., } r\text{-homogeneous of degree } m > -l \]

\[
\frac{\partial V(x)}{\partial x} g(x) = 0 \Rightarrow \sup_{v \in F(x)} L_v V < 0, \forall x \in \mathbb{R}^n \setminus \{0\}.
\]

\(r\)-homogeneous of degree 0 (discontinuous) Controllers

\[
\begin{align*}
  u &= -k \varphi_1 (x) = -k \left[ L_g(x)V(x) \right]^0, \\
  u &= -k \varphi_2 (x) = -k \frac{L_g(x)V(x)}{\|x\|^{l+m}_{r,p}},
\end{align*}
\]

If \( k \geq k^* \), \( x = 0 \) is GAS, and if \( l < 0 \) it is Finite-Time Stable.

\[
\|x\|_{r,p} = \left( |x_1|^{\frac{p}{r_1}} + \cdots + |x_n|^{\frac{p}{r_n}} \right)^{\frac{1}{p}}, \quad p \geq \max r_i, \quad \text{homogeneous norm}.
\]
HOSM design based on CLF

The Basic Uncertain System

\[ \sum_{DI} : \begin{cases} 
\dot{x}_i = x_{i+1}, \quad i = 1, \ldots, \rho - 1, \\
\dot{x}_\rho \in [-C, C] + [K_m, K_M]u. 
\end{cases} \]

- The Design is reduced to find a CLF.
- By a Back-Stepping-like procedure construct a CLF

Define

\[ \bar{x}^T_i = [x_1, \ldots, x_i], \]
\[ r = (r_1, r_2, \ldots, r_\rho) = (\rho, \rho - 1, \ldots, 1), \]
\[ \alpha_{\rho} \geq \alpha_{\rho - 1} \geq \cdots \geq \alpha_1 \geq \rho, \]
\[ m \geq r_i + \alpha_{i-1}. \]
Family of CLFs

**CLF r-homogeneous of degree** $m$: $\forall \gamma_i > 0$, $\exists k_i > 0$

$$V(x) = \gamma_r W_r(\bar{x}_r) + \cdots + \gamma_i W_i(\bar{x}_i) + \cdots + \gamma_1 \frac{\rho}{m} |x_1|^m$$

$$W_i(\bar{x}_i) = \frac{r_i}{m} |x_i|^{\frac{m}{r_i}} - \left[\nu_{i-1}\right]^{\frac{m-r_i}{r_i}} x_i + \left(1 - \frac{r_i}{m}\right) |\nu_{i-1}|^{\frac{m}{r_i}},$$

$$\nu_i(\bar{x}_i) = -k_i \left[\sigma_i\right]^{\frac{r_i+1}{\alpha_i}}, \quad \sigma_1 = \left[\bar{x}_1\right]^{\frac{\alpha_1}{r_1}}$$

$$\sigma_i(\bar{x}_i) = \left[\bar{x}_i\right]^{\frac{\alpha_i}{r_i}} - \left[\nu_{i-1}\right]^{\frac{\alpha_i}{r_i}} = \left[\bar{x}_i\right]^{\frac{\alpha_i}{r_i}} + k_i^{r_i} \left[\sigma_{i-1}\right]^{\frac{\alpha_i}{\alpha_{i-1}}},$$

$V(x)$ is a continuously differentiable and r-homogeneous CLF of degree $m$. 
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HOSM Controllers

**HOSM Discontinuous Controller**

Discontinuity at $\sigma_\rho(x) = 0$

$$u_D(x) = -k_\rho \left\| \sigma_\rho(x) \right\|^0, \ k_\rho \gg 0,$$

**HOSM Quasi-Continuous Controller**

Discontinuity only at $x = 0$

$$u_Q(x) = -k_\rho \frac{\sigma_\rho(x)}{M(x)}, \ k_\rho \gg 0,$$

$M(x)$ is any continuous r-homogeneous positive definite function of degree $\alpha_\rho$.

$\rho$-th order sliding mode $x = 0$ is established in Finite-Time.
Convergence Time Estimation

\[ T(x_0) \leq m\tau_\rho V_\rho^{\frac{1}{m}}(x_0), \]

where \( \tau_\rho \) is a function of the gains \((k_1, \ldots, k_\rho), K_m\) and \(C\).
Variable-Gain HOSM Controller

If $C = \bar{C} + \Theta(t, z)$, with constant $\bar{C}$ and time-varying $\Theta(t, z) \geq 0$ known.

**Variable-Gain Controller**

The Discontinuous Variable-Gain HOSM Controller

$$u_D(x) = -(K(t, z) + k_\rho) \left[\sigma_\rho(x)\right]^0, k_\rho \gg 0,$$

and the Quasi-Continuous Variable-Gain HOSM Controller

$$u_Q(x) = -(K(t, z) + k_\rho) \frac{\sigma_\rho(x)}{M(x)}, k_\rho \gg 0,$$

stabilize the origin $x = 0$ in Finite-Time if $k_\rho \gg 0$ and $K_m K(t, z) \geq \Theta(t, z)$. 
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For $\alpha_\rho \geq \cdots \geq \alpha_1 \geq \rho$, the discontinuous Controllers of orders $\rho = 2, 3, 4$ are given by

\begin{align*}
u_{2D} &= -k_2 \left[ [x_2]^{\alpha_2} + k_1^{\alpha_2} [x_1]^{\frac{\alpha_2}{2}} \right]^0 \\
u_{3D} &= -k_3 \left[ [x_3]^{\alpha_3} + k_2^{\alpha_3} \left[ [x_2]^{\frac{\alpha_2}{2}} + k_1^{\frac{\alpha_2}{2}} [x_1]^{\frac{\alpha_2}{3}} \right]^{\frac{\alpha_3}{\alpha_2}} \right]^0 \\
u_{4D} &= -k_4 \left[ [x_4]^{\alpha_4} + k_3^{\alpha_4} \left[ [x_3]^{\frac{\alpha_3}{2}} + k_2^{\frac{\alpha_3}{2}} \left[ [x_2]^{\frac{\alpha_2}{3}} + k_1^{\frac{\alpha_2}{3}} [x_1]^{\frac{\alpha_2}{4}} \right]^{\frac{\alpha_3}{\alpha_2}} \right]^{\frac{\alpha_4}{\alpha_3}} \right]^0
\end{align*}

and are, in general, of the type of the Nested Sliding Controllers (Levant 2005).
Discontinuous Relay Polynomial HOSM Controllers

Cruz-Zavala & Moreno (2014, 2016) [4, 3, 5]; Ding, Levant & Li (2015, 2016) [7, 8].

For \( \alpha_{\rho} = \alpha_{\rho-1} = \cdots = \alpha_1 = \alpha \geq \rho \) discontinuous ”relay polynomial” controllers

\[
\begin{align*}
    u_{2R} &= -k_2 \text{sign} \left( [x_2]^\alpha + \bar{k}_1 [x_1]^\frac{\alpha}{2} \right), \\
    u_{3R} &= -k_3 \text{sign} \left( [x_3]^\alpha + \bar{k}_2 [x_2]^\frac{\alpha}{2} + \bar{k}_1 [x_1]^\frac{\alpha}{3} \right), \\
    u_{4R} &= -k_4 \text{sign} \left( [x_4]^\alpha + \bar{k}_3 [x_3]^\frac{\alpha}{2} + \bar{k}_2 [x_2]^\frac{\alpha}{3} + \bar{k}_1 [x_1]^\frac{\alpha}{4} \right)
\end{align*}
\]

where for \( \rho = 2 \), \( \bar{k}_1 = k_1^\alpha \); for \( \rho = 3 \), \( \bar{k}_1 = k_2^\alpha k_1^\frac{\alpha}{2} \), \( \bar{k}_2 = k_2^\alpha \); and for general \( \rho \), \( \bar{k}_i = \prod_{j=i}^{\rho-1} k_j^{\frac{\alpha}{j}} \).
Quasi-Continuous Nested or Relay Polynomial HOSM Controllers

For arbitrary $\beta_i > 0$

\[
\begin{align*}
    u_{2Q} &= -k_2 \frac{\left[ x_2 \right]^{\alpha_2} + k_1^{\alpha_2} \left[ x_1 \right]^{\alpha_2} \beta_1}{\left| x_2 \right|^{\alpha_2} \beta_1 \left| x_1 \right|^{\alpha_2}}, \\
    u_{3Q} &= -k_3 \frac{\left[ x_3 \right]^{\alpha_3} + k_2^{\alpha_3} \left[ x_2 \right]^{\alpha_2} + k_1^{\alpha_2} \left[ x_1 \right]^{\alpha_2} \beta_2 \left| x_2 \right|^{\alpha_2} \beta_1 \left| x_1 \right|^{\alpha_2}}{\left| x_3 \right|^{\alpha_3} \beta_2 \left| x_2 \right|^{\alpha_2} \beta_1 \left| x_1 \right|^{\alpha_2}}, \\
    u_{4Q} &= -k_4 \frac{\left[ x_4 \right]^{\alpha_3} + k_3^{\alpha_3} \left[ x_3 \right]^{\alpha_3} + k_2^{\alpha_2} \left[ x_2 \right]^{\alpha_2} + k_1^{\alpha_2} \left[ x_1 \right]^{\alpha_2} \beta_3 \left| x_3 \right|^{\alpha_3} \beta_2 \left| x_2 \right|^{\alpha_2} \beta_1 \left| x_1 \right|^{\alpha_2}}{\left| x_4 \right|^{\alpha_4} \beta_3 \left| x_3 \right|^{\alpha_3} \beta_2 \left| x_2 \right|^{\alpha_2} \beta_1 \left| x_1 \right|^{\alpha_2}}.
\end{align*}
\]
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Numerical Gain Calculation

- $k_i$, for $i = 1, \cdots, \rho - 1$, selected to render $V(x)$ a CLF,
- $k_{\rho}$ to obtain $\dot{V} < 0$.
- Fix $m$, $\rho$, $\alpha_i$ and $\gamma_i$. Set $k_1 > 0$, for $i = 2, \cdots, \rho$

$$k_i > \max_{\bar{x}_i \in S_i} \{\Phi_i(\bar{x}_i)\} =: G_i(k_1, \cdots, k_{i-1}), \quad (2)$$

- Maximization feasible since

  1. $\Phi$ is $r$-homogeneous of degree 0: achieves all its values on the homogeneous unit sphere $S_i = \{\bar{x}_i \in \mathbb{R}^i : \|\bar{x}_i\|_{r,\rho} = 1\}$, and
  2. $\Phi$ is upper-semicontinuous $\Rightarrow$ it has a maximum on $S_i$.

- Parametrization

$$k_i = \mu_i k_1^{\rho-(i-1)}, \quad k_{\rho} > \frac{1}{K_m} (\mu_{\rho} k_1^\rho + C'), \quad (3)$$

for some positive constants $\mu_i$ independent of $k_1$.

- Parametrization can be used for all controllers, but $k_{\rho}$ different for discontinuous and quasi-continuous controllers.
Analytical Gain calculation

It is possible (but cumbersome) to provide for any order an analytical estimation of the values of the gains using classical inequalities.

The simplest case with \( \rho = 3, u_{3D}, u_{3R} \) and \( u_{3Q} \)

For any values of \( \alpha_3 \geq \alpha_2 \geq r_1 = 3, m \geq r_2 + \alpha_2, \gamma_1 > 0 \), \( 0 < \eta < 1 \) and \( k_1 > 0 \),

\[
k_2 > \frac{r_2 2^{\frac{m-2r_2}{\alpha_2}}}{m-1} \left( \frac{(m-r_1) 2^{\frac{\alpha_2-r_2}{\alpha_2}}}{m-1} \right)^{\frac{m-r_1}{r_2}} \left( \frac{\gamma_1 + \frac{m-r_2}{r_1} k_1^{\frac{m}{r_2}}}{\eta \gamma_1 k_1} \right)^{\frac{m-1}{r_2}}.
\]
Remarks

- By homogeneity the gain scaling with any $L \geq 1$

$$k^T = (k_1, \cdots, k_\rho) \rightarrow k_L^T = (L^{\frac{1}{\rho}} k_1, \cdots, L^{\frac{1}{\rho+1-i}} k_i, \cdots, Lk_\rho)$$

preserves the stability for any $\alpha_j$.

- Convergence will be accelerated for $L > 1$, or the size of the allowable perturbation $C$ will be incremented to $LC$.

- The gains obtained by means of the LF can be very large for practical applications, so that a simulation-based gain design is eventually necessary (see [5]).

- The gain design problem is an important and unexplored one.
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Example

Kinematic model of a car [10]

\[ \dot{z}_1 = v \cos(z_3), \quad \dot{z}_2 = v \sin(z_3), \quad \dot{z}_3 = \left( \frac{v}{L} \right) \tan(z_4), \quad \dot{z}_4 = u, \]

- \( z_1, z_2 \): cartesian coordinates of the rear-axle middle point,
- \( z_3 \): the orientation angle,
- \( z_4 \): the steering angle, (Actual control)
- \( v \): the longitudinal velocity (\( v = 10 \, \text{m.s}^{-1} \)),
- \( L \): distance between the two axles (\( L = 5 \, \text{m} \)), and
- \( u \): the control input.

\( u \) is used as a new control input in order to avoid discontinuities on \( z_4 \).
Control Objective

- **Control Task:** steer the car from a given initial position to the trajectory $z_{2\text{ref}} = 10 \sin (0.05 z_1) + 5$ in finite time.
- Turn on the controllers after Observer converged (0.5 sec.).
- Sliding variable $\sigma = z_2 - z_{2\text{ref}}$, relative degree $\rho = 3$, Model

$$\dot{x}_1 = x_2, \quad \dot{x}_2 = x_3, \quad \dot{x}_3 = \phi(\cdot) + \gamma(\cdot) u,$$

where $x = [\sigma \quad \dot{\sigma} \quad \ddot{\sigma}]^T$.
- Simulations: Euler’s method, sampling time $\tau = 0.0005$.
- Bounds: $|\phi| \leq C_0 = 49.63$, $K_m = 6.38 \leq \gamma \leq K_M = 46.77$.
- Controllers: (i) Levant’s Discontinuous Controller (L3) with $\beta_1 = 1$, $\beta_2 = 2$ and $k_3 = 20$; (ii) Levant’s Quasi-Continuous Controller (Q3) with $\beta_1 = 1$, $\beta_2 = 2$ and $k_3 = 24.5$; and (iii) Proposed Discontinuous Controller (E3) $u_{3D} = -k_3 [\sigma_3]^0$ with $k_1 = 1$, $k_2 = 1.5$, $k_3 = 20$. 

Lyapunov SMC  Jaime A. Moreno  UNAM  88
Figure: Left column: Levant’s Discontinuous Controller (L3), Middle Column: Levant’s Quasi-Continuous Controller (Q3); Right Column: Proposed Discontinuous Controller (E3).
Figure: Accuracy: (a) with (L3); (b) with (Q3); (c) with (E3); (d) with (Q3) ($k_3 = 70$).

Advantage of (E3): combines fast convergence rate of (L3) with smooth transient response of (Q3).
New: Methodological approach to design HOSM controllers using CLF.

Different alternatives to find CLFs: Back-stepping, Polynomial methods, ...

It can be extended to design controllers with Fixed-Time convergence.

Drawback: Calculation of gains $k_i$ needs maximization of 0-degree homogeneous functions.
Part III

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Basic Observation Problem

Variations of the observation problem: with unknown inputs, practical observers, robust observers, stochastic framework to deal with noises, ....
Consider a nonlinear system without inputs, $x \in \mathbb{R}^n$, $y \in \mathbb{R}$

$$\dot{x}(t) = f(x(t)) \quad , \quad x(t_0) = x_0$$

$$y(t) = h(x(t))$$

Differentiating the output

$$y(t) = h(x(t))$$

$$\dot{y}(t) = \frac{d}{dt} h(x(t)) = \frac{\partial h(x)}{\partial x} \dot{x}(t) = \frac{\partial h(x)}{\partial x} f(x) := L_f h(x)$$

$$\ddot{y}(t) = \frac{\partial L_f h(x)}{\partial x} \dot{x}(t) = \frac{\partial L_f h(x)}{\partial x} f(x) := L_f^2 h(x)$$

$$\vdots$$

$$y^{(n-1)}(t) = \frac{\partial L_f^{n-2} h(x)}{\partial x} \dot{x}(t) = \frac{\partial L_f^{n-2} h(x)}{\partial x} f(x) := L_f^{n-1} h(x)$$

where $L_f^k h(x)$ are Lie’s derivatives of $h$ along $f$. 
Evaluating at $t = 0$

$$
\begin{bmatrix}
  y(0) \\
  \dot{y}(0) \\
  \ddot{y}(0) \\
  \vdots \\
  y^{(k)}(0)
\end{bmatrix}
= 
\begin{bmatrix}
  h(x_0) \\
  L_f h(x_0) \\
  L_f^2 h(x_0) \\
  \vdots \\
  L_f^k h(x_0)
\end{bmatrix}
:= \mathcal{O}_n(x_0)
$$

$\mathcal{O}_n(x)$: Observability map

**Theorem**

*If $\mathcal{O}_n(x)$ is injective (invertible) $\rightarrow$ The NL system is observable.*
In the coordinates of the output and its derivatives

\[ z = O_n(x), \quad x = O_n^{-1}(z) \]

the system takes the (observability) form

\[ \dot{z}_1 = z_2 \]
\[ \dot{z}_2 = z_3 \]
\[ \vdots \]
\[ \dot{z}_n = \phi(z_1, z_2, \ldots, z_n) \]
\[ y = z_1 \]

So we can consider a system in this form as a basic structure.
A Simple Observer and its Properties

Plant: \[
\dot{x}_1 = x_2, \quad \dot{x}_2 = w(t)
\]
Observer: \[
\dot{\hat{x}}_1 = -l_1 (\hat{x}_1 - x_1) + \hat{x}_2, \quad \dot{\hat{x}}_2 = -l_2 (\hat{x}_1 - x_1)
\]
Estimation Error: \[
e_1 = \hat{x}_1 - x_1, \quad e_2 = \hat{x}_2 - x_2
\]
\[
\dot{e}_1 = -l_1 e_1 + e_2, \quad \dot{e}_2 = -l_2 e_1 - w(t)
\]

Figure: Linear Plant with an unknown input and a Linear Observer.
**Figure**: Behavior of Plant and the Linear Observer without unknown input.
Figure: Behavior of Plant and the Linear Observer with unknown input.
Figure: Behavior of Plant and the Linear Observer without UI with large initial conditions.
Figure: Behavior of Plant and the Linear Observer without UI with very large initial conditions.
Recapitulation.

Linear Observer for Linear Plant

- If no unknown inputs/Uncertainties: it converges exponentially fast.
- If there are unknown inputs/Uncertainties: no convergence. At best bounded error.
- Convergence time depends on the initial conditions of the observer.
- Is it possible to alleviate these drawbacks?
Recapitulation.

Linear Observer for Linear Plant

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Linear Observer for Linear Plant

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Is it possible to alleviate these drawbacks?
Sliding Mode Observer (SMO)

Figure: Linear Plant with an unknown input and a SM Observer.

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Figure: Behavior of Plant and the SM Observer without unknown input.
Figure: Behavior of Plant and the SM Observer with unknown input.
Recapitulation.

Sliding Mode Observer for Linear Plant

- If no unknown inputs/Uncertainties: $e_1$ converges in finite time, and $e_2$ converges exponentially fast.
- If there are unknown inputs/Uncertainties: no convergence. At best bounded error. Only $e_1$ converges in finite time!
- Convergence time depends on the initial conditions of the observer
- It is not the solution we expected! None of the objectives has been achieved!
Recapitulation.

Sliding Mode Observer for Linear Plant

- If no unknown inputs/Uncertainties: $e_1$ converges in finite time, and $e_2$ converges exponentially fast.
- If there are unknown inputs/Uncertainties: no convergence. At best bounded error. Only $e_1$ converges in finite time!
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- Convergence time depends on the initial conditions of the observer
  - It is not the solution we expected! None of the objectives has been achieved!
Recapitulation.

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- If there are unknown inputs/Uncertainties: no convergence. At best bounded error. Only $e_1$ converges in finite time!
- Convergence time depends on the initial conditions of the observer
- **It is not the solution we expected!** None of the objectives has been achieved!
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Super-Twisting Algorithm (STA)

Plant:

\[
\begin{align*}
\dot{x}_1 &= x_2, \\
\dot{x}_2 &= w(t)
\end{align*}
\]

Observer:

\[
\begin{align*}
\dot{\hat{x}}_1 &= -l_1 |e_1|^\frac{1}{2} \text{sign}(e_1) + \hat{x}_2, \\
\dot{\hat{x}}_2 &= -l_2 \text{sign}(e_1)
\end{align*}
\]

Estimation Error: \( e_1 = \hat{x}_1 - x_1, e_2 = \hat{x}_2 - x_2 \)

\[
\begin{align*}
\dot{e}_1 &= -l_1 |e_1|^\frac{1}{2} \text{sign}(e_1) + e_2 \\
\dot{e}_2 &= -l_2 \text{sign}(e_1) - w(t)
\end{align*}
\]

Solutions in the sense of Filippov.
Figure: Linear Plant with an unknown input and a SOSM Observer.
Figure: Behavior of Plant and the Non Linear Observer without unknown input.
\textbf{Figure:} Behavior of Plant and the Non Linear Observer with unknown input.
Figure: Behavior of Plant and the Non Linear Observer without UI with large initial conditions.
Recapitulation.

Super-Twisting Observer for Linear Plant

- If no unknown inputs/Uncertainties: $e_1$ and $e_2$ converge in finite-time!
- If there are unknown inputs/Uncertainties: $e_1$ and $e_2$ converge in finite-time! Observer is insensitive to perturbation/uncertainty!
- Convergence time depends on the initial conditions of the observer. This objective is not achieved!
Recapitulation.

Super-Twisting Observer for Linear Plant

- If no unknown inputs/Uncertainties: $e_1$ and $e_2$ converge in finite-time!
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Generalized Super-Twisting Algorithm (GSTA)

Plant:
\[
\begin{align*}
\dot{x}_1 &= x_2 , \\
\dot{x}_2 &= w(t) 
\end{align*}
\]

Observer:
\[
\begin{align*}
\dot{\hat{x}}_1 &= -l_1 \phi_1 (e_1) + \hat{x}_2 , \\
\dot{\hat{x}}_2 &= -l_2 \phi_2 (e_1) 
\end{align*}
\]

Estimation Error: \( e_1 = \hat{x}_1 - x_1, \ e_2 = \hat{x}_2 - x_2 \)
\[
\begin{align*}
\dot{e}_1 &= -l_1 \phi_1 (e_1) + e_2 \\
\dot{e}_2 &= -l_2 \phi_2 (e_1) - w(t) ,
\end{align*}
\]

Solutions in the sense of Filippov.

\[
\begin{align*}
\phi_1 (e_1) &= \mu_1 |e_1|^{\frac{1}{2}} \text{sign} (e_1) + \mu_2 |e_1|^{\frac{3}{2}} \text{sign} (e_1) , \ \mu_1 , \ \mu_2 \geq 0 , \\
\phi_2 (e_1) &= \frac{\mu_1^2}{2} \text{sign} (e_1) + 2 \mu_1 \mu_2 e_1 + \frac{3}{2} \mu_2^2 |e_1|^2 \text{sign} (e_1) ,
\end{align*}
\]
Figure: Linear Plant with an unknown input and a Non Linear Observer.
Figure: Behavior of Plant and the Non Linear Observer without unknown input and large initial conditions.
Figure: Behavior of Plant and the Non Linear Observer without unknown input and very large initial conditions.
Figure: Behavior of Plant and the Non Linear Observer with UI with large initial conditions.
Effect: Convergence time independent of I.C.

Figure: Convergence time when the initial condition grows.
Recapitulation.

Generalized Super-Twisting Observer for Linear Plant

- If no unknown inputs/Uncertainties: $e_1$ and $e_2$ converge in finite-time!
- If there are unknown inputs/Uncertainties: $e_1$ and $e_2$ converge in finite-time! Observer is insensitive to perturbation/uncertainty!
- Convergence time is independent of the initial conditions of the observer!
- All objectives were achieved!
- How to proof these properties?
Recapitulation.

Generalized Super-Twisting Observer for Linear Plant

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Recapitulation.

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- Convergence time is independent of the initial conditions of the observer!.

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- Convergence time is independent of the initial conditions of the observer!.
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How to proof these properties?
Recapitulation.

Generalized Super-Twisting Observer for Linear Plant

- If no unknown inputs/Uncertainties: $e_1$ and $e_2$ converge in finite-time!
- If there are unknown inputs/Uncertainties: $e_1$ and $e_2$ converge in finite-time! Observer is insensitive to perturbation/uncertainty!
- Convergence time is independent of the initial conditions of the observer!
- All objectives were achieved!
- How to proof these properties?
What have we achieved?

An algorithm

- Robust: it converges despite of unknown inputs/uncertainties
- Exact: it converges in finite-time
- The convergence time can be preassigned for any arbitrary initial condition.
- But there is no free lunch!

It is useful for

- Observation
- Estimation of perturbations/uncertainties
- Control: Nonlinear PI-Control
- in practice?

Some Generalizations are available but Still a lot is missing
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We propose a *Family* of strong Lyapunov functions, that are Quadratic-like.

This family allows the estimation of convergence time,

It allows to study the robustness of the algorithm to different kinds of perturbations,

All results are obtained in a Linear-Like framework, known from classical control,

The analysis can be obtained in the same manner for a linear algorithm, the classical ST algorithm and a combination of both algorithms (GSTA), that is non homogeneous.
Generalized STA

\[ \dot{x}_1 = -k_1 \phi_1 (x_1) + x_2 \]
\[ \dot{x}_2 = -k_2 \phi_2 (x_1) , \] 

Solutions in the sense of Filippov.

\[ \phi_1 (e_1) = \mu_1 |e_1|^{\frac{1}{2}} \text{sign} (e_1) + \mu_2 |e_1|^q \text{sign} (e_1) , \mu_1 , \mu_2 \geq 0 , q \geq 1 , \]
\[ \phi_2 (e_1) = \frac{\mu_1^2}{2} \text{sign} (e_1) + \left( q + \frac{1}{2} \right) \mu_1 \mu_2 |e_1|^{q-\frac{1}{2}} \text{sign} (e_1) + 
+ q\mu_2^2 |e_1|^{2q-1} \text{sign} (e_1) , \]

- **Standard STA:** $\mu_1 = 1$, $\mu_2 = 0$
- **Linear Algorithm:** $\mu_1 = 0$, $\mu_2 > 0$, $q = 1$.
- **GSTA:** $\mu_1 = 1$, $\mu_2 > 0$, $q > 1$.
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   ▪ Differentiation Problem
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   ▪ Examples
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Quadratic-like Lyapunov Functions

System can be written as:

\[ \dot{z} = \phi_1'(x_1) A z, \quad z = \begin{bmatrix} \phi_1(x_1) \\ x_2 \end{bmatrix}, \quad A = \begin{bmatrix} -k_1 & 1 \\ -k_2 & 0 \end{bmatrix}. \]

*Family of strong* Lyapunov Functions:

\[ V(x) = z^T P z, \quad P = P^T > 0. \]

Time derivative of Lyapunov Function:

\[ \dot{V}(x) = \phi_1'(x_1) z^T (A^T P + PA) z = -\phi_1'(x_1) z^T Q z \]

Algebraic Lyapunov Equation (ALE):

\[ A^T P + PA = -Q \]
Figure: The Lyapunov function.
Lyapunov Function

**Proposition**

- If $A$ Hurwitz then $x = 0$ Finite-Time stable (if $\mu_1 = 1$) and for every $Q = Q^T > 0$, $V(x) = \zeta^T P \zeta$ is a global, strong Lyapunov function, with $P = P^T > 0$ solution of the ALE, and

\[
\dot{V} \leq -\gamma_1 (Q, \mu_1) V^{\frac{1}{2}}(x) - \gamma_2 (Q, \mu_2) V(x),
\]

where

\[
\gamma_1 (Q, \mu_1) \triangleq \mu_1 \frac{\lambda_{\min} \{Q\} \lambda_{\min}^{\frac{1}{2}} \{P\}}{2\lambda_{\max} \{P\}}, \quad \gamma_2 (Q, \mu_2) \triangleq \mu_2 \frac{\lambda_{\min} \{Q\}}{\lambda_{\max} \{P\}}.
\]

- If $A$ is not Hurwitz then $x = 0$ unstable.
Proposition

If $k_1 > 0$, $k_2 > 0$, and $\mu_2 \geq 0$ a trajectory of the GSTA starting at $x_0 \in \mathbb{R}^2$ converges to the origin in finite time if $\mu_1 = 1$, and it reaches that point at most after a time

$$T = \begin{cases} 
\frac{2}{\gamma_1(Q,\mu_1)} V^{\frac{1}{2}} (x_0) & \text{if } \mu_2 = 0 \\
\frac{2}{\gamma_2(Q,\mu_2)} \ln \left( \frac{\gamma_2(Q,\mu_2)}{\gamma_1(Q,\mu_1)} V^{\frac{1}{2}} (x_0) + 1 \right) & \text{if } \mu_2 > 0
\end{cases}
$$

When $\mu_1 = 0$ the convergence is exponential.

For Design: $T$ depends on the gains!
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16 Conclusions
GSTA with perturbations: ARI

GSTA with time-varying and/or nonlinear perturbations

\[
\begin{align*}
\dot{x}_1 &= -k_1 \phi_1 (x_1) + x_2 \\
\dot{x}_2 &= -k_2 \phi_2 (x_1) + \rho(t, x)
\end{align*}
\]

Assume

\[
2 |\rho(t, x)| \leq \delta
\]

Analysis: The construction of Robust Lyapunov Functions can be done with the classical method of solving an Algebraic Ricatti Inequality (ARI), or equivalently, solving the LMI

\[
\begin{bmatrix}
A^T P + PA + \epsilon P + \delta^2 C^T C & PB \\
B^T P & -1
\end{bmatrix} \leq 0,
\]

where

\[
A = \begin{bmatrix}
-k_1 & 1 \\
-k_2 & 0
\end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\
1 \end{bmatrix}.
\]
Overview

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16 Conclusions
Differentiation

- Signal \( f(t) \) is a Lebesgue-measurable function on \([0, \infty)\).

- \( f(t) = f_0(t) + v(t) \): unknown
  - \( f_0(t) \), unknown base signal, \( n \)-times differentiable,
  - \( |f_0|^{(n)}(t) \leq L \), \( L \) known
  - \( |v(t)| \leq \eta \) uniformly bounded noise signal.

- Using: \( \varsigma_1 = f_0(t), \ldots, \varsigma_{i+1} = f_0^{(i)}(t) = \frac{d^i}{dt^i} f_0(t), \ i = 1, \ldots, n \),

- state representation of the base signal

\[
\begin{align*}
\dot{\varsigma}_i &= \varsigma_{i+1}, \ i = 1, \ldots, n-1, \\
\dot{\varsigma}_n &= f_0^{(n)}(t) \\
y &= \varsigma_1 + v
\end{align*}
\]

- Differentiator = Observer with (bounded) Unknown Input
Differentiation

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  \]
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Lyapunov SMC    Jaime A. Moreno    UNAM
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The Linear or High Gain Differentiator

\[
\begin{align*}
\dot{x}_i &= -k_i \frac{1}{\epsilon^i} (x_1 - f) + x_{i+1}, \\
\vdots & \quad i = 1, \cdots, n - 1 \\
\dot{x}_n &= -k_n \frac{1}{\epsilon^n} (x_1 - f),
\end{align*}
\]

- Smooth differentiator
- Quadratic Lyapunov Function
- Gain optimization
Trade off → Optimization

![Graph showing the relationship between the error bound and the high-gain parameter epsilon.](image)
Levant’s Robust and Exact Differentiator

\[
\dot{x}_i = -k_i \lambda^i n \left[ x_1 - f \right]^\frac{n-i}{n} + x_{i+1}, \\
\vdots \quad i = 1, \ldots, n-1 \\
\dot{x}_n = -k_n \lambda \left[ x_1 - f \right]^0,
\]

\[
[z]^p = |z|^p \text{sign}(z)
\]

- Levant 1998 (2nd order), 2003 (arbitrary order)
- Discontinuous: Filippov’s Differential Inclusion
- In the absence of noise it converges exactly in finite time.
- Basic for Higher Order Sliding Modes. Extensions: J.P. Barbot, Fridman, ....
- Convergence proof: Geometry and Homogeneity
- It provides Qualitative properties. No gain design method.
Levant’s Robust and Exact Differentiator

\[ \dot{x}_i = -k_i \lambda \frac{i}{n} \left[ x_1 - f \right]^{\frac{n-i}{n}} + x_{i+1}, \]
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\[\dot{x}_i = -k_i \lambda \frac{i}{n} [x_1 - f] \frac{n-i}{n} + x_{i+1},\]
\[\vdots \quad i = 1, \ldots, n-1\]
\[\dot{x}_n = -k_n \lambda [x_1 - f]^0,\]

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Levant’s Robust and Exact Differentiator

- No Lyapunov Function available for arbitrary order.
- For order $n = 2$:
  - Polyakov and Poznyak (2009).
  - Moreno and Osorio (2008, 2012) a non-smooth Lyapunov function
    \[ V(e) = \left[ \frac{e_1}{d_1} e_2 \right] P \left[ \frac{e_1}{d_1} e_2 \right]^T \]
    - It provides necessary and sufficient conditions.
    - Detailed analysis and (gain) design possible.
- For order $n = 3$:
No Lyapunov Function available for arbitrary order.

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$$V(e) = \begin{bmatrix} [e_1]^{\frac{1}{2}} & e_2 \end{bmatrix} P \begin{bmatrix} [e_1]^{\frac{1}{2}} & e_2 \end{bmatrix}^T$$

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    Nonlinear inequalities to solve.
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Homogeneous Differentiators

\[
\dot{x}_i = -k_i \frac{\lambda^i}{n} \left[ x_1 - f \right]^{\frac{r_i+1}{r_1}} + x_{i+1},
\]
\[
\vdots
\]
\[
\dot{x}_n = -k_n \lambda \left[ x_1 - f \right]^{\frac{r_n+1}{r_1}},
\]

\[
0 < r_{i+1} = r_i + d, \; i = 1, \ldots, n, \; r_n = 1, \; -1 \leq d \leq 0
\]

- For \( d = 0 \): Linear (HG) Differentiator (Khalil and Coauthors).
- For \( d = -1 \): Levant’s Differentiator. (1998, 2003, ...).
- For \(-1 < d \leq 0\) Differentiator is continuous.
- For \(-1 = d\) Differentiator is discontinuous, i.e. (Differential Inclusion).
Homogeneous Differentiators

\[
\dot{x}_i = -k_i \lambda \frac{i}{n} \left[ x_1 - f \right]^{\frac{r_{i+1}}{r_1}} + x_{i+1}, \\
\vdots \quad i = 1, \cdots, n - 1 \\
\dot{x}_n = -k_n \lambda \left[ x_1 - f \right]^{\frac{r_{n+1}}{r_1}},
\]

\[0 < r_{i+1} = r_i + d, \quad i = 1, \ldots, n, \quad r_n = 1, -1 \leq d \leq 0\]

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\[ \dot{x}_i = -k_i \lambda \frac{i}{n} \left[ x_1 - f \right] \frac{r_i + 1}{r_1} + x_{i+1}, \]
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0 < r_{i+1} = r_i + d, i = 1, \ldots, n, r_n = 1, -1 \leq d \leq 0

- For d = 0: Linear (HG) Differentiator (Khalil and Coauthors).
- For d = −1: Levant’s Differentiator. (1998, 2003,...).
- For −1 < d ≤ 0 Differentiator is continuous
- For −1 = d Differentiator is discontinuous, i.e. (Differential Inclusion).
Homogeneous Differentiators

\[ \dot{x}_i = -k_i \lambda^{i/n} \left[ x_1 - f \right]^{r_{i+1}/r_1} + x_{i+1}, \]
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The differentiation error

Differentiation error \( e_i \triangleq x_i - f_0^{(i-1)} \)

\[
\begin{align*}
\dot{e}_i &= -k_i \lambda \frac{i}{n} \left[ e_1 + \nu \right] \frac{r_{i+1}}{r_1} + e_{i+1}, \\
\vdots & \quad i = 1, \cdots, n - 1 \\
\dot{e}_n &= -k_n \lambda \left[ e_1 + \nu \right] \frac{r_{n+1}}{r_1} - f^{(n)}(t),
\end{align*}
\]

- If \( f^{(n)}(t) \equiv 0 \) and \(-1 < d \leq 0\) homogeneous with homogeneity degree \( d \) and weights \( r = [r_1, \cdots, r_n] \).
- If \( f^{(n)}(t) \in [-L, L] \) and \( d = -1 \) is a homogeneous DI with homogeneity degree \( d = -1 \) and weights \( r = [n, n - 1, \cdots, 1] \).
- Family parametrized by degree of homogeneity \(-1 \leq d \leq 0\).
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Differentiation error $e_i \triangleq x_i - f_0^{(i-1)}$

\[
\dot{e}_i = -k_i \lambda \frac{i}{n} \left[ e_1 + v \right]^{\frac{r_{i+1}}{r_1}} + e_{i+1}, \quad i = 1, \ldots, n-1
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Properties of homogeneous differentiators

Properties

Set $\lambda = 1$. $\exists k_i(L, d)$ such that as $t \to \infty$ the differentiation error $e_i$

- For $v(t) \equiv 0$ and polynomial signals, $S^n_p = \{ f^{(n)}(t) \equiv 0 \}$
  - For $d = 0$: converges exponentially,
  - For $-1 \leq d < 0$: converges in finite time.
- For $v(t) \equiv 0$ and $n$-Lipschitz signals $S^n_L = \{ |f^{(n)}(t)| \leq L \}$
- For a uniformly bounded noise ($|v(t)| \leq \eta$) and $n$-Lipschitz signals $S^n_L = \{ |f^{(n)}(t)| \leq L \}$,
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Remarks

- For $d = 0$ this is well known. Lyapunov and Frequency analysis is possible.
- For $-\epsilon < d < 0$ Perruquetti et al. (2008) show convergence for Hurwitz gains.
- For continuous cases $-1 < d \leq 0$ there exist smooth Lyapunov functions: Yang and Lin (2004), Qian and Lin (2005), Andrieu et al. (2006, 2008, 2009), ...
- We extend the approach to the discontinuous case $d = -1$.
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The Lyapunov Function I

Fix $p \geq r_1 + r_2 = 2 - (2n - 3) d > 1$ and define

$$z_i = \frac{e_i}{k_{i-1}}, \quad \tilde{k}_i = \frac{k_i}{k_{i-1}}, \quad k_0 = 1, \quad \delta (t) = - \frac{f^{(n)}(t)}{k_{n-1}}.$$ 

$$Z_i(z_i, z_{i+1}) = \frac{r_i}{p} |z_i|^\frac{p}{r_i} - z_i \left[ z_{i+1} \right]^{\frac{p-r_i}{r_{i+1}}} + \left( \frac{p-r_i}{p} \right) |z_{i+1}|^{\frac{p}{r_{i+1}}},$$

$Z_i$ are continuously differentiable, positive semidefinite and $Z_i(z_i, z_{i+1}) = 0$ if and only if $\left[ z_i \right]^{\frac{p}{r_i}} = \left[ z_{i+1} \right]^{\frac{p}{r_{i+1}}}.$
Lyapunov Function

For every \( p \geq 2 - (2n - 3) d > 1 \) and \( \beta_i > 0 \) each differentiator of the family \(-1 \leq d \leq 0\) admits a strong, proper, smooth and \( r \)-homogeneous of degree \( p \) Lyapunov function of the form

\[
V(z) = \sum_{j=1}^{n-1} \beta_j Z_j (z_j, z_{j+1}) + \beta_n \frac{1}{p} |z_n|^p
\]

\( \beta_i > 0, i = 1, \ldots, n \).

- \( V(z) \) is positive definite and (due to homogeneity) radially unbounded.
- For the linear case \((d = 0, p = 2)\) \( V \) is a quadratic form.
Idea of the Proof

- The basic idea (similar to Yang and Lin 2004, Qian and Lin 2005, Andrieu et al. 2006, 2008, ...) is to reduce stepwise the observer and showing convergence for a smaller observer.

- For the discontinuous case an issue: $\dot{V}$ is discontinuous. Properties of continuous homogeneous functions are not valid. Two ways out

- The derivative satisfies

$$\dot{V} \leq -\alpha_2 V^{\frac{p+d}{p}} + \alpha_3 LV^{\frac{p-1}{p}} + \sum_{i=1}^{n} \tilde{k}_i \mu_i V^{\frac{p-r_i}{p}} |\eta|^{\frac{r_i+1}{r_1}},$$

for some $\mu_i > 0$.

- Using standard arguments: differentiation error is ISS with respect to the noise $\eta(t)$ and $f_0^n(t)$. 

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Some observations

- **Noise influence:**
  - For $d = -1$: $\exists \mu_i(k_i)$, s.t.
    $$|e_i| = |x_i - f_{0}^{i-1}| \leq \mu_i |\eta|^{\frac{n-i+1}{n}}$$
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- Gain selection: From down upwards ($k_n, k_{n-1}, \cdots, k_1$) independent of the order. Calculation by maximization of a homogeneous function.
- Parameter $\lambda > 1$ accelerates convergence and increases the allowed bound $L$, but it also increases the noise effect.
- Convergence Time Estimation
  $$\dot{V} \leq -\kappa V(z)^{\frac{p-1}{p}}, \kappa > 0, T(z_0) \leq \frac{p}{\kappa} V^{\frac{1}{p}}(z_0).$$
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    $$|e_i| = |x_i - f_0^{i-1}| \leq \mu_{i0}|\eta|$$

- Gain selection: From down upwards ($k_n, k_{n-1}, \ldots, k_1$) independent of the order. Calculation by maximization of a homogeneous function.
- Parameter $\lambda > 1$ accelerates convergence and increases the allowed bound $L$, but it also increases the noise effect.
- Convergence Time Estimation
  $$\dot{V} \leq -\kappa V(z)^{\frac{p-1}{p}}, \kappa > 0, \, T(z_0) \leq \frac{p}{\kappa} V^{\frac{1}{p}}(z_0).$$
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First Order differentiator \((n = 2)\)

\[
\begin{align*}
\dot{x}_1 &= -k_1 \left[ x_1 - f \right]^{\frac{1}{1-d}} + x_2 \\
\dot{x}_2 &= -k_2 \left[ x_1 - f \right]^{\frac{1+d}{1-d}} ,
\end{align*}
\]

- \(x_2(t) \approx f^{(1)}_0\).
- Homogeneous of degree is \(-1 \leq d \leq 0\), weights \(r_1 = 1 - d\), \(r_2 = 1\), and \(r_3 = 1 + d\).
- For \(d = -1\) Levant’s robust and exact differentiator, for \(d = 0\) linear differentiator.
Lyapunov Function is

\[
V(z_1, z_2) = \frac{1 - d}{2 - d} |z_1|^{\frac{2-d}{1-d}} - z_1 z_2 + \left(\frac{\beta + 1}{2 - d}\right) |z_2|^{2-d},
\]

Derivative \( \dot{V} \)

\[
\dot{V} = -k_1 |\sigma_1|^2 + \tilde{k}_2 (1 + \beta) s_1 \left[ z_1 \right]^{\frac{1+d}{1-d}} - \tilde{k}_2 \beta |z_1|^{\frac{2}{1-d}}
\]

\[
\sigma_1 = \left( \left[ z_1 \right]^{\frac{1}{1-d}} - z_2 \right), \quad s_1 = \left( z_1 - \left[ z_2 \right]^{1-d} \right)
\]
Required value of $k_1$ satisfies

$$\frac{k_1}{k_2} = \frac{k_1^2}{k_2} > \omega_2 \triangleq \max_{z \in \mathbb{R}^2} g_2(z_1, z_2),$$

$$g_2(z_1, z_2) \triangleq \frac{\left(s_1 - \beta \left[ z_2 \right]^{1-d} \right) \left[ z_1 \right]^{1+d}}{|\sigma_1|^2}.$$ 

$g_2(z_1, z_2)$ homogeneous of degree zero, upper semicontinuous and has a maximum, achieved on the homogeneous sphere.
Second Order differentiator \((n = 3)\)

\[
\begin{align*}
\dot{x}_1 &= -k_1 \left[ x_1 - f \right]^{\frac{1-d}{1-2d}} + x_2 \\
\dot{x}_2 &= -k_2 \left[ x_1 - f \right]^{\frac{1}{1-2d}} + x_3 \\
\dot{x}_3 &= -k_3 \left[ x_1 - f \right]^{\frac{1+d}{1-2d}}
\end{align*}
\]

\[x_2 (t) \approx f^{(1)} (t) \text{ and } x_3 (t) \approx f^{(2)} (t).\]

\[
V (z) = Z_1 (z_1, z_2) + \beta_2 Z_2 (z_2, z_3) + \frac{\beta_3}{p} |z_3|^p
\]

\[
\frac{k_2^2}{k_1 k_3} > \omega_{23}, \quad \frac{k_1 k_2}{k_3} > \omega_{13},
\]
Simulations

- Signal $f_0(t) = 0.5 \sin(0.5t) + 0.5 \cos(t)$,
- Bounded derivative $\left| f_0^{(3)}(t) \right| \leq 1$.
- Simulations for linear $d = 0$, homogeneous $d = -0.5$ and Levant’s $d = -1$ differentiators.
- Noise $v(t) = \varepsilon \sin(\omega t)$, $\varepsilon = 0.001$, and $\omega = 1000$.
- Gains $k_1 = 3$, $k_2 = 1.5\sqrt{3}$, $k_3 = 1.1$.
- Euler-method with step size $\tau = 3 \times 10^{-4}$.
Simulations

- Signal $f_0(t) = 0.5 \sin(0.5t) + 0.5 \cos(t)$,
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Simulations

- Signal \( f_0(t) = 0.5 \sin(0.5t) + 0.5 \cos(t) \),
- Bounded derivative \( \left| f_0^{(3)}(t) \right| \leq 1 \).
- Simulations for linear \( d = 0 \), homogeneous \( d = -0.5 \) and Levant’s \( d = -1 \) differentiators.
- Noise \( v(t) = \varepsilon \sin(\omega t) \), \( \varepsilon = 0.001 \), and \( \omega = 1000 \).
- Gains \( k_1 = 3 \), \( k_2 = 1.5\sqrt{3} \), \( k_3 = 1.1 \).
- Euler-method with step size \( \tau = 3 \times 10^{-4} \).
Simulations: No noise

\( f_0(t) = 0.5 \sin(0.5t) + 0.5 \cos(t), \text{ noise}=0 \)
Simulations: Noisy measurement

\[ f_0(t) = 0.5 \sin(0.5t) + 0.5 \cos(t), \text{ noise} = 0.001 \sin(1000t) \]
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  - But the set of stabilizing gains is not covered.
  - Coefficients for the noise and High derivative effect can be calculated (conservative).
  - Comparison is possible (future work).
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Part IV

Construction of Lyapunov Functions using Generalized Forms
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Generalized Forms (GF) method [Sanchez and Moreno 2014, 2016]

**Basic Idea:**
Transform a PDE

\[
\frac{\partial V(x)}{\partial x} f(x) = -W(x)
\]

⇒ Algebraic equation + Positive Definiteness

**Motivation: Lyapunov functions for LTI systems**

System: \( \dot{x} = Ax, \ x \in \mathbb{R}^n \),

LF Candidate: \( V(x) = x^T Px \),

LF Derivative: \( -\dot{V} = W(x) = x^T Q x \),

Algebraic Lyapunov Equation: \( PA + A^T P = -Q \).
Classic forms:

### Homogeneous polynomial of degree $m \in \mathbb{Z}_{\geq 0}$

$$f(x) = \sum_{j=1}^{M} \alpha_j \prod_{i=1}^{n} x_i^{\rho_{i,j}}, \quad \rho_{i,j} \in \mathbb{Z}_{\geq 0}, \quad \sum_{i=1}^{n} \rho_{i,j} = m$$

Finite $M \in \mathbb{Z}_{>0}$, $\alpha_j \in \mathbb{R}$, $x \in \mathbb{R}^n$.

**Example**

$$f(x) = 2x_1^4 + 3x_1^3x_2 + 5x_1^2x_2^2 - 6x_1x_2^3 + 8x_2^4.$$
Generalized forms (GF):

Homogeneous function of degree $m \in \mathbb{R}_{\geq 0}$ with weights $\mathbf{r}$

$$f(x) = \sum_{j=1}^{N} \alpha_j \prod_{i=1}^{n} v_{i,j}(x_i, \rho_{i,j}), \quad \rho_{i,j} \in \mathbb{R}_{\geq 0}, \quad \sum_{i=1}^{n} r_i \rho_{i,j} = m$$

$$v_{i,j}(x_i, \rho_{i,j}) = |x_i|^{\rho_{i,j}}, \quad [x_i]^{\rho_{i,j}}$$

Finite $N \in \mathbb{Z}_{>0}$, $\alpha_j \in \mathbb{R}$, $x \in \mathbb{R}^n$.

Example

$$f(x) = \kappa_1 |x_1|^{\frac{5\pi}{2}} + \kappa_2 \left[ x_1 \right]^{\frac{\pi}{2}} |x_2|^{\frac{4\pi}{3}}, \quad \kappa_i \in \mathbb{R}.$$  

$m = 5\pi$, $\mathbf{r} = [2, 3]^\top$. 

Classic forms $\subset$ GFs
Homogeneous polynomial system ($\kappa = 2$, $r = [1, 3]^{\top}$)

$\Sigma : \dot{x}_1 = -x_1^3 + x_2, \quad \dot{x}_2 = -x_1^5,$

Weak Lyapunov function [Bacciotti & Rosier, 2005]

$V(x) = \frac{1}{6}x_1^6 + \frac{1}{2}x_2^2, \quad \dot{V} = -x_1^8,$

Theorem [Sanchez, 2016]

For $\Sigma$, there is no strict LF in the class of homogeneous polynomials of any degree $m$ for any weights $r$.

Strict Lyapunov function for $\Sigma$

$V(x) = \alpha_1 x_1^6 - \alpha_{12} x_1 \left[x_2\right]^{\frac{5}{3}} + \alpha_2 x_2^2,$ (GF!)
GF systems: HOSM

Twisting algorithm [Levant, 1993]

\[
\dot{x}_1 = x_2, \quad \dot{x}_2 = -k_1 [x_1]^{0} - k_2 [x_2]^{0},
\]

Super–Twisting algorithm [Levant, 1993]

\[
\dot{x}_1 = -k_1 [x_1]^{\frac{1}{2}} + x_2, \quad \dot{x}_2 = -k_2 [x_1]^{0},
\]

CTA [Torres et al., 2013]

\[
\begin{align*}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= -k_1 [x_1]^{\frac{1}{3}} - k_2 [x_2]^{\frac{1}{2}} + x_3 \\
\dot{x}_3 &= -k_3 [x_1]^{0} - k_4 [x_2]^{0}
\end{align*}
\]

\[
[\cdot]^\rho = \text{sign}(\cdot)|\cdot|^\rho
\]
More GF systems

Continuous homogeneous systems (Finite time)

\[ \dot{x}_1 = x_2, \quad \dot{x}_2 = -k_1 \left[ x_1 \right]^\frac{1}{2} - k_2 \left[ x_2 \right]^\frac{2}{3}, \quad \kappa = -1, \quad \mathbf{r} = [3, 2]^\top \]

 Polynomial homogeneous systems

\[ \dot{x}_1 = -x_1^3 + x_2, \quad \dot{x}_2 = -x_1^5, \quad \kappa = 2, \quad \mathbf{r} = [1, 3]^\top \]

Linear systems

\[ \dot{x} = Ax, \quad x \in \mathbb{R}^n, \quad \kappa = 0, \quad \mathbf{r} = [1, \ldots, 1]^\top \]
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GFs properties

Theorem

1. Sums of GFs (degree $m$, weights $r$) are GFs of degree $m$.
2. $F_a$ and $F_b$ GF of degree $m_a$ and $m_b$, with weights $r$, $F_aF_b$ is a GF of degree $m_a + m_b$.

Theorem

1. A GF is differentiable almost everywhere (coordinate hyperplanes).
2. A continuous GF is differentiable everywhere if its exponents $\rho_{i,j} \neq 0$ are such that

   $$\rho_{i,j} \geq 1, \text{ if } v_i(x_i, \rho_{i,j}) = \left[ x_i \right]^{\rho_{i,j}}$$

   $$\rho_{i,j} > 1, \text{ if } v_i(x_i, \rho_{i,j}) = |x_i|^{\rho_{i,j}}, \forall i, j.$$  

3. Partial derivatives of a differentiable GF are GFs.
GFs properties

Corollary

- $\dot{x} = f(x), \ x \in \mathbb{R}^n$, is a GF system of degree $\kappa$ with $\mathbf{r}$.
- $V : \mathbb{R}^n \to \mathbb{R}$ is a GF of degree $m$ with weights $\mathbf{r}$.

$\Rightarrow W(x) = -\nabla V(x) \cdot f(x)$ is GF of degree $\bar{m} = m + \kappa$.

Structure for positive definiteness

$$V(x, \alpha) = \sum_{i=1}^{n} \alpha_i |x_i|^m r_i + \sum_{j=1}^{q} \bar{\alpha}_j \prod_{i=1}^{n} \varphi_{i,j}(x_i, \rho_{i,j}),$$

$$W(x, \beta) = \sum_{i=1}^{n} \beta_i |x_i|^\bar{m} r_i + \sum_{j=1}^{\bar{q}} \bar{\beta}_j \prod_{i=1}^{n} \bar{\varphi}_{i,j}(x_i, \bar{\rho}_{i,j}),$$
Polynomial characterization

Commensurable exponents: $\rho_i / \rho_j \in \mathbb{Q}$

Isomorphism: $d^\gamma : \mathcal{P}_n \rightarrow \mathcal{D}_\gamma$

\[ d^\gamma(y) = [\sigma_1 y_1^{\mu_1}, \ldots, \sigma_n y_n^{\mu_n}]^\top, \mu_i \in \mathbb{Q}_{>0} \]

Hyperoctants: $\mathcal{D}_\gamma \subset \mathbb{R}^n$. $\sigma_i = \text{sign}(x_i)$, $x \in \mathcal{D}_\gamma$

$\mathcal{P}_n = \{z \in \mathbb{R}^n \mid z_i > 0, i = 1, 2, \ldots, n\}$.

Lemma

If $f : \mathbb{R}^n \rightarrow \mathbb{R}$ is GF of degree $m$ with weights $\mathbf{r}$ and rational exponents, then there exist $\mu_i \in \mathbb{Q}_{>0}$ such that every $f_{\mathcal{D}_\gamma} \circ d^\gamma : \mathcal{P}_n \rightarrow \mathbb{R}$ is a form.

Associated forms of a GF $f$

\[ f(x) : \{f_1(y), \ldots, f_{2^n}(y)\}, \quad f_i : \overline{\mathcal{P}}_n \rightarrow \mathbb{R} \]
Polynomial characterization

Isomorphism:

\[ d^1(y) \quad d^2(y) \quad d^3(y) \quad d^4(y) \]

Lemma

A GF \( f : \mathbb{R}^n \to \mathbb{R} \) is positive definite if its associated forms are positive definite.
Example

GF

\[ V(x) = |x_1|^5 + x_1 x_2 + |x_2|^5, \quad m = 5, \quad r = [3, 2]^T \]

Isomorphism

\[ d^\gamma(z) = [\sigma_1 z_1^3, \sigma_2 z_2^2]^T \]

\[ V_\gamma = V \circ d^\gamma : \tilde{\mathcal{P}} \to \mathbb{R} \]

- \( \tilde{\mathcal{D}}_1 = \{ x_1 \geq 0, x_2 \geq 0 \}, \quad V_1(z) = z_1^5 + z_1^3 z_2 + z_2^5 \)
- \( \tilde{\mathcal{D}}_2 = \{ x_1 \leq 0, x_2 \geq 0 \}, \quad V_2(z) = z_1^5 - z_1^3 z_2 + z_2^5 \)
- \( \tilde{\mathcal{D}}_3 = \{ x_1 \leq 0, x_2 \leq 0 \}, \quad V_3(z) = z_1^5 + z_1^3 z_2 + z_2^5 \)
- \( \tilde{\mathcal{D}}_4 = \{ x_1 \geq 0, x_2 \leq 0 \}, \quad V_4(z) = z_1^5 - z_1^3 z_2 + z_2^5 \)

\[ V(x) : \{ V_1(z), V_2(z), V_3(z), V_4(z) \} \]
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Theorem (Pólya, 1928)

The (classic) Form $F : \mathcal{P}_n \setminus \{0\} \rightarrow \mathbb{R}$, is positive if and only if there exists $p_0 \in \mathbb{N}$ such that for all $p \geq p_0$, the coefficients of the form

$$G(z) = (z_1 + z_2 + \cdots + z_n)^p F(z), \quad \forall z \in \mathcal{P}_n \setminus \{0\},$$

are positive.

Example

$$V(z) = \alpha_1 z_1^3 - \alpha_2 z_1^2 z_2 + \alpha_3 z_2^3, \quad \alpha_i > 0,$$

$$G_p(z) = (z_1 + z_2)^p V(z),$$
Pólya’s Theorem

\[ p = 1 \]

\[ G_1(z) = \alpha_1 z_1^4 + (\alpha_1 - \alpha_2) z_1^3 z_2 - \alpha_2 z_1^2 z_2^2 + \alpha_3 z_1 z_2^3 + \alpha_3 z_2^4. \]

\[ p = 2 \]

\[ G_2(z) = \alpha_1 z_1^5 + (2\alpha_1 - \alpha_2) z_1^4 z_2 + (\alpha_3 - \alpha_2) z_1^2 z_2^3 + (\alpha_1 - 2\alpha_2) z_1^3 z_2^2 + 2\alpha_3 z_1 z_2^4 + \alpha_3 z_2^5. \]

Inequalities

\[ \alpha_1 > 0, \ 2\alpha_1 - \alpha_2 \geq 0, \ \alpha_3 - \alpha_2 \geq 0, \ \alpha_1 - 2\alpha_2 \geq 0, \ \alpha_3 > 0, \]
Pólya’s Theorem

System of linear inequalities $A_v \alpha \succeq 0$

$$A_v = \begin{bmatrix} 1 & 0 & 0 & 2 & 0 & 1 \\ 0 & 1 & 0 & -1 & -1 & -2 \\ 0 & 0 & 1 & 0 & 1 & 0 \end{bmatrix}^\top, \quad \alpha = [\alpha_1 \: \alpha_2 \: \alpha_3]^\top.$$ 

Polyhedral cone

$$C = \{\alpha \in \mathbb{R}^d : A\alpha \succeq 0\}$$

Minkowski-Weyl

$$C = \{\alpha = B\gamma : 0 \leq \gamma \in \mathbb{R}^q\}$$

Software: Skeleton [Zolotykh, 2012]
Sum of Squares (SOS) representation: Hilbert’s 17th prob.

The (classic) Form $F$ of degree $2q$ is positive semi-definite if

$$F(z) = \sum_{i=1}^{N} (f_i)^2.$$

Example

$$F(z) = z_1^2 + 2z_1 z_2 + z_2^2 = (z_1 + z_2)^2,$$

Example

$$F(z) = z_1^6 - 2z_1^4 z_2 z_3 + z_1^2 z_2^4 + z_1^2 z_2^2 z_3^2 - 2z_1 z_2 z_3 + z_3^6$$

$$= (z_1^3 - z_1 z_2 z_3)^2 + (z_3^3 - z_1 z_2^2)^2.$$
SOS representation

**SOS-Quadratic form [Choi et al., 1995]**

\[ F(z) : SOS \iff F(z) = y(z)^T P y(z), \]

\[ F = F(z; \alpha) , \text{ LMI problem: } P(\alpha) \succeq 0 \]


**Positive definiteness**

\[ \tilde{F}(z) = F(z) - \epsilon \sum_{i=1}^{n} z_i^m, \quad \epsilon \in \mathbb{R}_{>0} \]
Positive definite GFs

Let $F = F(x; \alpha)$ be a GF and \{${F_i}$\} its associated forms.

Pólya’s Theorem

$$G_p(z; \alpha) = (z_1 + z_2 + \cdots + z_n)^p F_i(z; \alpha),$$

Linear inequalities: $A_i \alpha > 0$,

SOS representation

$$\bar{F}_i(z; \alpha) = F_i(z; \alpha) - \epsilon \sum_{i=1}^{n} z_i^m, \quad \epsilon \in \mathbb{R}_{>0}$$

LMIs: $P_i(\alpha) \geq 0$.

Adequate isomorphism.
Pólya

- Necessary and Sufficient condition for positive definiteness
- Leads to Linear Inequalities (not LMIs)
- The complete solution for a given power $p$ can be completely characterized
- Available Software (e.g., Skeleton)

SOS

- A Sufficient condition for positive definiteness
- Leads to LMIs
- Available Software (e.g., SOSTOOLS)
- Allows to include objective functions (optimization)
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GF Lyapunov function

GF system:
\[ \dot{x} = f(x; k) \]

GF LF candidate:
\[ V(x; \alpha) \]

Derivative:
\[ W(x; \beta) \]

Associated forms
\[ \{ V_i(z; \alpha), W_i(z; \beta) \} \]

\[ \beta \text{ bilinear} \]
\[ \beta = \beta(\alpha, k), \quad \beta = M(k)\alpha, \quad \beta = \tilde{M}(\alpha)k \]
Algorithm.  I

Given a GF system $\dot{x} = f(x; k)$ of degree $\kappa$ with weights $r$, 

**Step 1** Chose some terms $v_{ij}(x_i, \rho_i)$ in

$$V(x, \alpha) = \sum_{i=1}^{n} \alpha_i |x_i|^{r_i} + \sum_{j=1}^{q} \bar{\alpha}_j \prod_{i=1}^{n} v_{i,j}(x_i, \rho_{i,j})$$

**Step 2** Take the derivative of $V$ along the trajectories of the system and obtain

$$W(x, \beta) = \sum_{i=1}^{n} \beta_i |x_i|^{\bar{m}_i} + \sum_{j=1}^{\bar{q}} \bar{\beta}_j \prod_{i=1}^{n} \bar{v}_{i,j}(x_i, \bar{\rho}_{i,j})$$
Algorithm. II

Step 3 Considering

- Homogeneity: $\sum_{i=1}^{n} r_i \rho_{i,j} = m$
  \[
  m > \max_i \{r_i\}
  \]
- Differentiability:
  \[
  \begin{align*}
  \rho_{i,j} &\geq 1, \text{ for } v_i(x_i, \rho_{i,j}) = |x_i|^{\rho_{i,j}} \\
  \rho_{i,j} &> 1, \text{ for } v_i(x_i, \rho_{i,j}) = |x_i|^{\rho_{i,j}}
  \end{align*}
  \]

Restrict the exponents $\rho_{i,j}$ and the signs of $\bar{\alpha}_j$ such that the coefficients $\beta_i$ can be strictly positive. If not, go back to Step 1 and increase $q$ or change $v_{ij}(x_i, \rho_i)$.

Step 4 Set $m$ and $\rho_{i,j}$.

Step 5 Chose $\mu_i$ in: $d^\gamma(y) = [\sigma_1 y_1^{\mu_1}, \ldots, \sigma_n y_n^{\mu_n}]^\top$
Algorithm. III

Step 6  Compute the associated forms

\{V_1, \ldots, V_{2n}\}, \quad \{W_1, \ldots, W_{2n}\}

Step 7  Find $\alpha$ and $k$ for positive definiteness of $V_i, W_i$

- Solving Pólya’s inequalities
- Finding SOS representation

Bilinear problem!
### Analysis ($k$ given):

#### Pólya’s procedure

\[
\{ AV_i \alpha \succ 0, AW_i \beta \succ 0 \}, \quad \beta = M(k)\alpha
\]

Solve for $\alpha$ the system of linear inequalities:

\[
\{ AV_i \alpha \succ 0, AW_i M(k)\alpha \succ 0 \}
\]

#### SOS procedure

Define the forms:

\[
\tilde{V}_j(y) = V_j(y) - \epsilon \sum_{i=1}^{n} y_i^\delta, \quad \tilde{W}_j(y) = W_j(y) - \epsilon \sum_{i=1}^{n} y_i^{\bar{\delta}},
\]

$\delta, \bar{\delta}$, degrees of $V_i, W_i$. Solve for $\alpha$ the system of LMIs:

\[
\{ P_{\tilde{V}_i}(\alpha) \geq 0, P_{\tilde{W}_i}(\alpha) \geq 0 \}
Pólya’s procedure

\[ \{ A_V \alpha \succ 0, \ A_W \beta \succ 0 \}, \quad \beta = \bar{M}(\alpha)k \]

- Solve for \( \alpha \) the system \( \{ A_V \alpha \succ 0 \} \) and choose an \( \alpha^* \)
- Solve for \( k \) the system \( \{ A_W \bar{M}(\alpha^*)k \succ 0 \} \)

SOS procedure

\[ \{ P_{\tilde{V}}(\alpha) \geq 0, \ P_{\tilde{W}}(\alpha, k) \geq 0 \} \]

- Solve for \( \alpha \) the LMIs \( \{ P_{\tilde{V}}(\alpha) \geq 0 \} \) and choose an \( \alpha^* \)
- Solve for \( k \) the LMIs \( \{ P_{\tilde{W}}(\alpha^*, k) \geq 0 \} \)
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Super–Twisting I

Super–Twisting algorithm [Levant, 1993]

\[
\dot{x}_1 = -k_1 \left| x_1 \right|^\frac{1}{2} + x_2 , \quad \dot{x}_2 = -k_2 \left| x_1 \right|^0 ,
\]

Homogeneous of degree \( \kappa = -1 \) with weights \( \mathbf{r} = [2, 1]^\top \).

Lyapunov function candidate

\[
V(x) = \alpha_1 |x_1|^\frac{m}{2} + \bar{\alpha}_2 \left| x_1 \right|^\rho_1 \left| x_2 \right|^\rho_2 + \alpha_3 |x_2|^m .
\]

Homogeneity \( \rho_2 = m - 2\rho_1 \).

Necessary conditions for positive definiteness: \( \alpha_1, \alpha_3 > 0 \).

Differentiability: \( m > 2, \rho_1 \geq 1 \) and \( \rho_2 = m - 2\rho_1 \geq 1 \).

Choosing \( m = 3 \)

\[
V(x) = \alpha_1 |x_1|^\frac{3}{2} + \bar{\alpha}_2 \left| x_1 \right|^\rho_1 \left| x_2 \right|^{3-2\rho_1} + \alpha_3 |x_2|^3 .
\]
\[ \dot{V} = -W(x) \]

\[ W(x) = \frac{3\alpha_1 k_1}{2} |x_1| - \frac{3\alpha_1}{2} \left[ x_1 \right]^\frac{1}{2} x_2 + 3\alpha_3 k_2 \left[ x_2 \right]^2 \left[ x_1 \right]^0 + \]

\[ \bar{\alpha}_2 k_2 (3 - 2\rho_1) |x_1|^{\rho_1} |x_2|^{2-2\rho_1} + \bar{\alpha}_2 k_2 \rho_1 \left[ x_1 \right]^{\rho_1-\frac{1}{2}} \left[ x_2 \right]^{3-2\rho_1} \]

\[ -\bar{\alpha}_2 \rho_1 |x_1|^{\rho_1-1} |x_2|^{2-2\rho_1}. \]

\[ \rho_1 = 1 \text{ and } -\bar{\alpha}_2 = \alpha_2 > 0. \]

**LF Candidate**

\[ V(x) = \alpha_1 |x_1|^\frac{3}{2} - \alpha_2 x_1 x_2 + \alpha_3 |x_2|^3, \]

\[ W(x) = \beta_1 |x_1| - \beta_2 \left[ x_1 \right]^\frac{1}{2} x_2 + \beta_3 |x_2|^2 + \beta_4 \left[ x_1 \right]^0 |x_2|^2, \]
**Super–Twisting III**

### Coefficients of the Derivative

\[
\begin{align*}
\beta_1 &= \frac{3}{2} \alpha_1 k_1 - \alpha_2 k_2, \\
\beta_2 &= \frac{3}{2} \alpha_1 + \alpha_2 k_1, \\
\beta_3 &= \alpha_2, \\
\beta_4 &= 3 \alpha_3 k_2
\end{align*}
\]

**Note:** \( \beta_i \) is linear in \( \alpha_j \) and linear in \( k_j \) but not in both.

### LF conditions

Find \( \alpha_i, k_i \) so that \( V > 0 \) and \( W > 0 \).

### Isomorphism

\[
d'(z) = [\sigma_1 z_1^2, \sigma_2 z_2]^T
\]
\( V_\gamma = V \circ d^\gamma : \mathcal{P} \rightarrow \mathbb{R} \)

- \( \mathcal{D}_1 = \{ x_1 \geq 0, x_2 \geq 0 \} \), \( V_1(z) = \alpha_1 z_1^3 - \alpha_2 z_1^2 z_2 + \alpha_3 z_2^2 \)
- \( \mathcal{D}_2 = \{ x_1 \leq 0, x_2 \geq 0 \} \), \( V_2(z) = \alpha_1 z_1^3 + \alpha_2 z_1^2 z_2 + \alpha_3 z_2^2 \)
- \( \mathcal{D}_3 = \{ x_1 \leq 0, x_2 \leq 0 \} \), \( V_3(z) = \alpha_1 z_1^3 + \alpha_2 z_1^2 z_2 + \alpha_3 z_2^2 \)
- \( \mathcal{D}_4 = \{ x_1 \geq 0, x_2 \leq 0 \} \), \( V_4(z) = \alpha_1 z_1^3 - \alpha_2 z_1^2 z_2 + \alpha_3 z_2^2 \)

\( W_\gamma = W \circ d^\gamma : \mathcal{P} \rightarrow \mathbb{R} \)

- \( \mathcal{D}_1 = \{ x_1 \geq 0, x_2 \geq 0 \} \), \( W_1(z) = \beta_1 z_1^2 - \beta_2 z_1 z_2 + (\beta_3 + \beta_4) z_2^2 \)
- \( \mathcal{D}_2 = \{ x_1 \leq 0, x_2 \geq 0 \} \), \( W_2(z) = \beta_1 z_1^2 + \beta_2 z_1 z_2 + (\beta_3 - \beta_4) z_2^2 \)
- \( \mathcal{D}_3 = \{ x_1 \leq 0, x_2 \leq 0 \} \), \( W_3(z) = \beta_1 z_1^2 + \beta_2 z_1 z_2 + (\beta_3 - \beta_4) z_2^2 \)
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\[ \{x_1x_2 < 0\}: z_1 \geq 0, z_2 \geq 0 \]

\[
V(z) = \alpha_1 z_1^3 + \alpha_2 z_1^2 z_2 + \alpha_3 z_2^3, \\
W(z) = \beta_1 z_1^2 + \beta_2 z_1 z_2 + (\beta_3 - \beta_4) z_2^2.
\]

\[ \beta_3 > \beta_4. \]

\[ \{x_1x_2 \geq 0\}: z_1 \geq 0, z_2 \geq 0 \]

\[
V(z) = \alpha_1 z_1^3 - \alpha_2 z_1^2 z_2 + \alpha_3 z_2^3, \\
W(z) = \beta_1 z_1^2 - \beta_2 z_1 z_2 + (\beta_3 + \beta_4) z_2^2.
\]

Just these forms must be analysed!

\((V \text{ was analysed in the example of Pólya's theorem})\)
Super-Twisting, Pólya’s procedure

Fix $\alpha = [2.1, 1, 1.1]^{\top}$,

$$G_2(z) = (z_1 + z_2)^p W(z) \Rightarrow A_w \beta > 0,$$

**Double description**

$$A_w M(\alpha)[1 \ k^{\top}]^{\top} > 0 \iff k = B_w \gamma,$$

$\gamma \in \mathbb{R}^q, \gamma_i > 0, \sum_{i=1}^q \gamma_i = 1, q$ is the number of columns of $B_w$.

Solution for $p = 6B_w = \begin{bmatrix} 3.788 & 2.325 & 3.019 \\ 0.303 & 0.303 & 0.257 \end{bmatrix}$,

For example, with $\gamma = (1/3)[1, 1, 1]^{\top}$

$$k_1 = 3.04, \ k_2 = 0.28$$
Super-Twisting, SOS procedure I

Change of variables: $z_1 > 0, z_2 > 0, y_1, y_2 \in \mathbb{R}$

$$(z_1, z_2) \mapsto (y_1^2, y_2^2).$$

Classical Forms of even degree: $y \in \mathbb{R}^2$

$$V(y) = \alpha_1 y_1^6 - \alpha_2 y_1^4 y_2^2 + \alpha_3 y_2^6,$$

$$W(y) = \beta_1 y_1^4 - \beta_2 y_1^2 y_2^2 + (\beta_3 + \beta_4) y_2^4.$$

SOS $\Rightarrow$ LMI [Parrilo, 2000]

$$\tilde{V}(y) = V(y) - \epsilon(y_1^6 + y_2^6) > 0, \quad \epsilon > 0.$$

$$\tilde{V}(y) = \psi^T(y)Q_v \psi(y), \quad \psi(y) = [y_1^3, y_1^2 y_2, y_1 y_2^2, y_2^3]^T.$$
\[ Q_v = \begin{bmatrix} \bar{\alpha}_1 & 0 & -\lambda_1 & 0 \\ 0 & 2\lambda_1 \alpha_1 \alpha_0 - \alpha_2 & 0 & -\lambda_2 \\ -\lambda_1 & 0 & \alpha_0 - 2\lambda_2 \alpha_2 \bar{\alpha}_3 & 0 \\ 0 & -\lambda_2 & 0 & \bar{\alpha}_3 \end{bmatrix} > 0. \]
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HOSM Differentiator

\[ f(t) = f_0(t) + \nu(t), \quad \left| f_0^{(n)}(t) \right| \leq L \]

Levant’s Differentiator

\[
\begin{align*}
\dot{x}_i &= -k_i \left[ x_1 - f \right]^{n-i} + x_{i+1}, i = 1, \ldots, n - 1 \\
\dot{x}_n &= -k_n \left[ x_1 - f \right]^0.
\end{align*}
\]

Dynamics of the Differentiation error: \( z_i = \frac{x_i - f^{(i-1)}_0}{k_{i-1}} \)

\[
\begin{align*}
\dot{z}_i &= -\tilde{k}_i \left( [z_1 + \nu]^{n-i} - z_{i+1} \right), \quad \tilde{k}_i = \frac{k_i}{k_{i-1}}, \\
\dot{z}_n &= -\tilde{k}_n \left[ z_1 + \nu \right]^0 - \frac{f_0^{(n)}(t)}{k_{n-1}}.
\end{align*}
\]

Homogeneous: degree \( d = -1 \), weights \( r = (n, n - 1, \ldots, 1) \).
Generalized Form as Lyapunov Function

**LF:** for $p \geq 2n - 1$ and any $\beta_i > 0$

$$V(z) = \sum_{j=1}^{n-1} \beta_j Z_j(z_j, z_{j+1}) + \beta_n \frac{1}{p} |z_n|^p, \beta_i > 0$$

$$Z_i(z_i, z_{i+1}) = \frac{n+1-i}{p} |z_i|^{\frac{p}{n+1-i}} +$$

$$- z_i \left[ z_{i+1} \right]^{\frac{p-n-1+i}{n-i}} + \left( \frac{p-n-1+i}{p} \right) |z_{i+1}|^{\frac{p}{n-i}}.$$

**Convergence Time Estimation**

$$\dot{V} \leq -\kappa V(z) \frac{p-1}{p}, \kappa > 0$$

$$T(z_0) \leq \frac{p}{\kappa} V(z_0)^{\frac{1}{p}}(z_0).$$
Gains calculation by SOS

Gain Calculation using SOS and \( p = 2n - 1 \)

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<th>( k_1 )</th>
<th>( k_2 )</th>
<th>( k_3 )</th>
<th>( k_4 )</th>
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<td>4.95</td>
<td>1.03</td>
<td>–</td>
<td>1</td>
</tr>
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<td>5.81</td>
<td>17.75</td>
<td>15.45</td>
<td>1.02</td>
<td>1</td>
</tr>
</tbody>
</table>
## Summary

### Pros
- Provides a computable way to calculate LFs for a fairly general class of homogeneous systems.
- It can be extended to non homogeneous systems.
- The algebraic problem to solve is a system of linear inequalities (Pólya) or an LMI (SOS). It is linear in the coefficients of the LF candidate and in the gains.

### Cons
- Restricted to ”polynomial” systems
- Course of high $p$ for Pólya and SOS.
## Summary

### Pros
- Provides a computable way to calculate LFs for a fairly general class of homogeneous systems.
- It can be extended to non homogeneous systems.
- The algebraic problem to solve is a system of linear inequalities (Pólya) or an LMI (SOS). It is linear in the coefficients of the LF candidate and in the gains.

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- Restricted to ”polynomial” systems
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Part V

Continuous HOSM Controllers
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27 $r = 2$ Continuous Twisting Controller
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29 Discontinuous Integral Controller: State Feedback
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   • Caveat: Lack of Homogeneity
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Perturbed second-order plant

\[ \dot{x}_1 = x_2 \\
\dot{x}_2 = u + \mu(t), \]

- Discontinuous controller (SOSM), e.g. Twisting controller, rejects \textit{bounded} perturbation,
- strong \textit{chattering},
- Precision
  \[ |x_1| \leq \nu_1 \tau^2, \quad |x_2| \leq \nu_2 \tau. \]
- Chattering reduction requires continuous control signal.
Chattering Attenuation: Standard

\[ \begin{align*}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= x_3 \triangleq u + \mu(t) \\
\dot{x}_3 &= \dot{u} + \dot{\mu}(t) \\
\dot{u} &= k_3 \vartheta_2(x_1, x_2, x_3)
\end{align*} \]

Properties

- Levant 2003
- Continuous control signal \( u(t) \Rightarrow \text{chattering attenuation} \)
- Rejects \textbf{Lipschitz} continuous (possibly unbounded) perturbation,
- Precision \( |x_1| \leq \nu_1 \tau^3, \quad |x_2| \leq \nu_2 \tau^2, \quad |x_3| \leq \nu_3 \tau \)
- \textbf{Drawback:} It requires \((x_1, x_2)\) and \(\dot{x}_2\)!
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Relative degree $r = 1$, the Super-Twisting System:

- **System:**
  \[
  \dot{x}_1 = u + \rho(x, t)
  \]

- **Discontinuous Control:**
  \[
  u = -k \text{ sign}(x_1)
  \]

- **Discontinuous Integral Control (Super-Twisting):**
  \[
  u = -k_1 |x_1|^{\frac{1}{2}} \text{ sign}(x_1) + z
  \]
  \[
  \dot{z} = -k_2 \text{ sign}(x_1)
  \]

- **Closed Loop System:**
  \[
  \dot{x}_1 = -k_1 |x_1|^{\frac{1}{2}} \text{ sign}(x_1) + x_2
  \]
  \[
  \dot{x}_2 = -k_2 \text{ sign}(x_1) + \dot{\rho}(x, t)
  \]
Relative degree $r = 1$, the Super-Twisting System:

- System:
  \[ \dot{x}_1 = u + \rho(x, t) \]

- Discontinuous Control:
  \[ u = -k \, \text{sign}(x_1) \]

- Discontinuous Integral Control (Super-Twisting):
  \[
  \begin{align*}
  u &= -k_1 |x_1|^\frac{1}{2} \, \text{sign}(x_1) + z \\
  \dot{z} &= -k_2 \, \text{sign}(x_1)
  \end{align*}
  \]

- Closed Loop System:
  \[
  \begin{align*}
  \dot{x}_1 &= -k_1 |x_1|^\frac{1}{2} \, \text{sign}(x_1) + x_2 \\
  \dot{x}_2 &= -k_2 \, \text{sign}(x_1) + \dot{\rho}(x, t)
  \end{align*}
  \]
Relative degree $r = 1$, the Super-Twisting System:

- System:
  \[ \dot{x}_1 = u + \rho(x, t) \]

- Discontinuous Control:
  \[ u = -k \ \text{sign}(x_1) \]

- Discontinuous Integral Control (Super-Twisting):
  \[
  \begin{align*}
  u &= -k_1 |x_1|^{1/2} \ \text{sign}(x_1) + z \\
  \dot{z} &= -k_2 \ \text{sign}(x_1)
  \end{align*}
  \]

- Closed Loop System:
  \[
  \begin{align*}
  \dot{x}_1 &= -k_1 |x_1|^{1/2} \ \text{sign}(x_1) + x_2 \\
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  \end{align*}
  \]
Relative degree $r = 1$, the Super-Twisting System:

- System:
  \[ \dot{x}_1 = u + \rho(x, t) \]

- Discontinuous Control:
  \[ u = -k \, \text{sign}(x_1) \]

- Discontinuous Integral Control (Super-Twisting):
  \[ u = -k_1 |x_1|^{\frac{1}{2}} \, \text{sign}(x_1) + z \]
  \[ \dot{z} = -k_2 \, \text{sign}(x_1) \]

- Closed Loop System:
  \[ \dot{x}_1 = -k_1 |x_1|^{\frac{1}{2}} \, \text{sign}(x_1) + x_2 \]
  \[ \dot{x}_2 = -k_2 \, \text{sign}(x_1) + \dot{\rho}(x, t) \]
Figure: State Trajectory with $\rho(t) = 0.5 \sin(t) + 0.25 \sin(2t)$

- Robust stabilization in finite time
- Continuous control signal
Figure: Super-Twisting Control Signal
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Continuous Terminal Sliding Mode Controller

\[ u = -k_1 L^{\frac{2}{3}} \left[ \phi_L (x_1, x_2) \right]^\frac{1}{3} + z \]
\[ \dot{z} = -k_2 L \left[ \phi_L (x_1, x_2) \right]^0 \]

\[ \phi_L (x_1, x_2) = x_1 + \frac{\alpha}{L^\frac{1}{2}} \left[ x_2 \right]^\frac{3}{2}, k_i > 0, L > 0. \]

Stability proof: Lyapunov function

\[ V (x) = \beta |x_1|^\frac{5}{3} + x_1 x_2 + \frac{2}{5} \alpha |x_2|^\frac{5}{2} - \frac{1}{k_1^3} x_2^3 x_3^3 + \gamma |x_3|^5, \]
\[ x_3 \triangleq z + \mu \]
Properties

- Kamal, Moreno, Chalanga, Bandyopadhyay, Fridman (2016).
- Continuous control signal $u(t) \Rightarrow$ chattering attenuation
- It rejects Lipschitz continuous (possibly unbounded) perturbation,
- Precision

$$|x_1| \leq \nu_1 \tau^3, \quad |x_2| \leq \nu_2 \tau^2, \quad |x_3| \leq \nu_3 \tau$$

- **Advantage:** It only requires $(x_1, x_2)$ and not $\dot{x}_2$!
- Estimation of the perturbation: $z(t) \to \mu(t)$.
Gain calculation by function maximization

Table: Sets of gain values obtained by maximization for $L = 1$. 

<table>
<thead>
<tr>
<th>Set</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>$k_1$</td>
<td>4.4</td>
<td>4.5</td>
<td>7.5</td>
<td>16</td>
</tr>
<tr>
<td>$k_2$</td>
<td>2.5</td>
<td>2</td>
<td>2</td>
<td>7</td>
</tr>
<tr>
<td>$\alpha$</td>
<td>20</td>
<td>28.7</td>
<td>7.7</td>
<td>1</td>
</tr>
<tr>
<td>$\Delta$</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
</tbody>
</table>
Phase portrait of $x_1$ and $x_2$

Figure: Phaseportrait
Phaseportrait: Twisting-like behavior

Figure: Phaseportrait
Overview

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Continuous Twisting Algorithm

\[ u(x) = -k_1|x_1|^{\frac{1}{3}} - k_2|x_2|^{\frac{1}{2}} + z \]
\[ \dot{z} = -k_3|x_1|^0 - k_4|x_2|^0 \]

**Stability proof: Lyapunov function**

\[ V(x) = \alpha_1|x_1|^\frac{5}{3} + \alpha_2x_1x_2 + \alpha_3|x_2|^\frac{5}{2} + \alpha_4x_1|x_3|^2 - \alpha_5x_2x_3^3 + \alpha_6|x_3|^5. \]
\[ x_3 \triangleq z + \mu \]
Properties

- Torres, Sanchez, Fridman, Moreno (2015), [25]
- Continuous control signal $u(t) \Rightarrow$ chattering attenuation
- It rejects Lipschitz continuous (possibly unbounded) perturbation,
- Precision

$$|x_1| \leq \nu_1 \tau^3, \quad |x_2| \leq \nu_2 \tau^2, \quad |x_3| \leq \nu_3 \tau$$

- Advantage: It only requires $(x_1, x_2)$ and not $\dot{x}_2$!
- Estimation of the perturbation: $z(t) \to \mu(t)$.
- Gain calculation using Polya’s Theorem.
- Convergence Time estimation

$$T_c \leq \frac{5}{\gamma} V^{\frac{1}{5}} (x(0))$$
Virtues of Continuous HOSM

- Continuous control signal $\Rightarrow$ chattering attenuation.
- Extension to arbitrary order

\[
\begin{align*}
\dot{x}_i &= x_{i+1}, \ i = 1, \ldots, \rho - 1, \\
\dot{x}_\rho &= -k_1 \phi(x) + z + \mu(t) \\
\dot{z} &= -k_2 [\phi(x)]^0
\end{align*}
\]

- Rejects Lipschitz (possibly unbounded) continuous perturbations versus bounded perturbations of HOSM.
- Requires only $x$ and not $\dot{x}_\rho$.
- Lyapunov approach also extended for arbitrary order systems.
- Interesting approach from (Chitour, Harmouche, Laghrouche).
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Continuous Integral Controller (PID)

- System

\[
\begin{align*}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= u + \rho(t),
\end{align*}
\]

- PID-Controller (e.g. linear)

\[
\begin{align*}
u &= -k_1(x_1, x_2) + k_I(x_3) \\
\dot{x}_3 &= -k_2(x_1, x_2),
\end{align*}
\]

- \(k_{1,2}(x_1, x_2)\) continuous, \(k_I(x_3)\) continuous/discontinuous.

- Constant perturbations/references \(\Rightarrow\) Asymptotic convergence and insensitive to perturbation!

- Arbitrary perturbations/ref \(\Rightarrow\) Practical convergence.

- Estimation of \(\rho(t)\) is not required for implementation.

- More general: Internal Model Principle based controller.
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Lyapunov SMC   Jaime A. Moreno   UNAM
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- Combine Integral Action and Discontinuous Control.
- $k_1(x_1, x_2)$ and $k_I(x_3)$ continuous, $k_2(x_1, x_2)$ discontinuous.
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- Continuous control signal $\Rightarrow$ Chattering reduction.
- For simplicity (!?) we add Homogeneity.
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Continuous and Homogeneous State Feedback Controller

\[ u = -k_1 \left| x_1 \right|^{\frac{1}{3}} - k_2 \left| x_2 \right|^{\frac{1}{2}} \]

Closed Loop System:

\[ \dot{x}_1 = x_2 \]
\[ \dot{x}_2 = -k_1 \left| x_1 \right|^{\frac{1}{3}} - k_2 \left| x_2 \right|^{\frac{1}{2}} + \rho(t) , \]

Lyapunov Function:

\[ V(x_1, x_2, x_3) = \gamma_1 |x_1|^{\frac{5}{3}} + \gamma_1 x_1 x_2 + |x_2|^{\frac{5}{2}} , \]

Sensitive to perturbations.
Homogeneous Integral + State Feedback Controller

Discontinuous Integral Controller \((k_1, k_2, k_3 > 0, k_4 \in \mathbb{R})\)

\[
\begin{align*}
u &= -k_1 \, x_1^{\frac{1}{3}} - k_2 \, x_2^{\frac{1}{2}} + z \\
\dot{z} &= -k_3 \left[ x_1 + k_4 \, x_2^{\frac{3}{2}} \right]^0
\end{align*}
\]

Closed Loop System:

\[
\begin{align*}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= -k_1 \, x_1^{\frac{1}{3}} - k_2 \, x_2^{\frac{1}{2}} + z + \rho(t), \\
\dot{z} &= -k_3 \left[ x_1 + k_4 \, x_2^{\frac{3}{2}} \right]^0
\end{align*}
\]
Homogeneous Integral + State Feedback Controller

Discontinuous Integral Controller \((k_1, k_2, k_3 > 0, k_4 \in \mathbb{R}, L > 0)\)

\[
\begin{align*}
    u &= -k_1 L^2 x_1^{\frac{2}{3}} - k_2 L^{\frac{1}{2}} x_2^{\frac{1}{2}} + z \\
    \dot{z} &= -k_3 L x_1 + k_4 L^{-\frac{3}{2}} x_2^{\frac{3}{2}} \\
    \dot{x}_1 &= x_2 \\
    \dot{x}_2 &= -k_1 L^2 x_1^{\frac{2}{3}} - k_2 L^{\frac{1}{2}} x_2^{\frac{1}{2}} + z + \rho(t), \\
    \dot{z} &= -k_3 L x_1 + k_4 L^{-\frac{3}{2}} x_2^{\frac{3}{2}} \\
\end{align*}
\]

Closed Loop System:

\(L > 0\) scaling gain:

If \(\rho(t) = 0\): Stability for \(L = 1 \Rightarrow\) Stability for any \(L > 0\).
Remarks

- **In contrast to the continuous Integral Controller:**
  - It tracks exactly, in finite time and robustly
  - arbitrary references with bounded $\ddot{r}(t)$
  - despite arbitrary (time) Lipschitz
    perturbations/uncertainties, i.e. $\|\dot{r}(t)\| \leq \Delta$, $\Delta$ constant
  - without an Internal Model.
- For implementation: $r(t)$ and $\dot{r}(t)$ are required but not $\ddot{r}(t)$.
- Define $x_3 = z + \rho$. After convergence $\Rightarrow x(t) = 0 \Rightarrow$
  $z(t) = -\rho(t)$: Integral action estimates the perturbation!
- Control signal is continuous $\Rightarrow$ Chattering attenuation.
- Gain selection:
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  - Set $k_1, k_2$ so that state feedback stable and well-behaved
  - Select $k_3 < 0$, $k_4 > 0$, $k_4 < 0$.
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    perturbations/uncertainties.
Remarks

- In contrast to the continuous Integral Controller:
  - It tracks exactly, in finite time and robustly
  - arbitrary references with bounded $\ddot{r}(t)$
  - despite arbitrary (time) Lipschitz
    perturbations/uncertainties, i.e. $\|\dot{r}(t)\| \leq \Delta$, $\Delta$ constant
  - without an Internal Model.
- For implementation: $r(t)$ and $\dot{r}(t)$ are required but not $\ddot{r}(t)$.
- Define $x_3 = z + \rho$. After convergence $\Rightarrow x(t) = 0 \Rightarrow z(t) = -\rho(t)$: Integral action estimates the perturbation!
- Control signal is continuous $\Rightarrow$ Chattering attenuation.
- Gain selection:
  - Set $k_1, k_2$ so that state feedback stable and well-behaved
    without perturbation.
  - Select $k_4 = 0$, $k_4 > 0$, $k_4 < 0$.
  - Select $k_3$ small to assure stability.
  - Select $L$ sufficiently large to compensate the
    perturbations/uncertainties.
A similar algorithm is the ”Continuous Twisting Algorithm”. The proof is based on a Generalized Forms technique.

\[ u = -k_1 \left[ x_1 \right]^{\frac{1}{3}} - k_2 \left[ x_2 \right]^{\frac{1}{2}} + z \]
\[ \dot{z} = -k_3 \left[ x_1 \right]^0 - k_4 \left[ x_2 \right]^0 \]

The ”High-Order Super Twisting”

\[ u = -k_1 \left[ x_2 + k_2 \left[ x_1 \right]^{\frac{2}{3}} \right]^{\frac{1}{2}} + z \]
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\end{align*}
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   - Lyapunov Function
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Homogeneous and smooth Lyapunov Function

Homogeneous and smooth Lyapunov Function \((L = 1)\)

\[
V_1 (x_1, x_2; x_3) = k_1 \left( \frac{3}{4} |x_1|^\frac{4}{3} - x_1 \frac{x_3}{k_1} + \frac{1}{4} \left| \frac{x_3}{k_1} \right|^4 \right) + \frac{1}{2} x_2^2 ,
\]

\[
V (x_1, x_2, x_3) = \gamma V_1^{\frac{5}{4}} (x_1, x_2; x_3) + \left( x_1 - \left( \frac{x_3}{k_1} \right)^3 \right) x_2 + \frac{\mu}{5} |x_3|^5 .
\]

Its derivative is given by

\[
\dot{V} (x_1, x_2, x_3) \in -W_1 (x_1, x_2; x_3) - k_3 W_2 (x_1, x_2; x_3)
\]
The Lyapunov function fulfills following differential inequality
\[ \dot{V}(x) \leq -\kappa V^{\frac{4}{5}}(x), \]
for some \( \kappa > 0 \) depending on the gains and \( \Delta \).

It implies robust finite time stability.

Convergence time estimation:
\[ T(x_0) \leq \frac{5}{\kappa} V^{\frac{1}{5}}(x_0). \]
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Caveat

Alternative \textit{Integral + state feedback} controllers:

- Linear Integral + state feedback controller (Homogeneous)
  \begin{align*}
  u & = -k_1 x_1 - k_2 x_2 + x_3 \\
  \dot{x}_3 & = -k_3 x_1
  \end{align*}

- Linear state feedback + Discontinuous Integral controller (Not Homogeneous)
  \begin{align*}
  u & = -k_1 x_1 - k_2 x_2 + x_3 \\
  \dot{x}_3 & = -k_3 \text{sign}(x_1)
  \end{align*}

- Discontinuous I-Controller (Extended Super-Twisting) (Homogeneous)
  \begin{align*}
  u & = -k_1 |x_1|^\frac{1}{3} \text{sign}(x_1) - k_2 |x_2|^\frac{1}{2} \text{sign}(x_2) + x_3 \\
  \dot{x}_3 & = -k_3 \text{sign}(x_1)
  \end{align*}
Controller without perturbation

Linear Integral controller

Linear+Discontinuous Integrator

Super-Twisting controller

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Controller with perturbation

- **Linear Integral controller**
  - $x_1$, $x_2$, and $u$ graphs over time $t$

- **Linear+Discontinuous Integrator**
  - $x_1$, $x_2$, and $u$ graphs over time $t$

- **Super-Twisting controller**
  - $x_1$, $x_2$, and $u$ graphs over time $t$
Remarks

- Linear stabilizes exponentially and is not insensitive to perturbation.
- Linear + Discontinuous Integrator causes oscillations (Harmonic Balance). This is structural and for any $n > 2$. Eliminated by Homogeneity.
- Extended ST: Convergence in finite time and insensitive to perturbations.
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Homogeneous Output Feedback Controller

Homogeneous State Feedback Controller + Homogeneous Observer

\[
\begin{align*}
\dot{x}_1 &= -l_1 \left[ \hat{x}_1 - x_1 \right]^\frac{2}{3} + \hat{x}_2 \\
\dot{x}_2 &= -l_2 \left[ \hat{x}_1 - x_1 \right]^\frac{1}{3} - k_1 \left[ x_1 \right]^\frac{1}{3} - k_2 \left[ \hat{x}_2 \right]^\frac{1}{2} \\
u &= -k_1 \left[ x_1 \right]^\frac{1}{3} - k_2 \left[ \hat{x}_2 \right]^\frac{1}{2}.
\end{align*}
\]

Homogeneous Integral + Output Feedback Controller

\[
\begin{align*}
\dot{x}_1 &= -l_1 \left[ \hat{x}_1 - x_1 \right]^\frac{2}{3} + \hat{x}_2 \\
\dot{x}_2 &= -l_2 \left[ \hat{x}_1 - x_1 \right]^\frac{1}{3} - k_1 \left[ x_1 \right]^\frac{1}{3} - k_2 \left[ \hat{x}_2 \right]^\frac{1}{2} \\
u &= -k_1 \left[ x_1 \right]^\frac{1}{3} - k_2 \left[ \hat{x}_2 \right]^\frac{1}{2} + z \\
\dot{z} &= -k_3 \left[ x_1 + k_4 \left[ \hat{x}_2 \right]^\frac{3}{2} \right]^0.
\end{align*}
\]
Simulations

We have implemented three controllers:

- A State Feedback (SF) controller with discontinuous integral term, with gains $k_1 = 2$, $k_2 = 5$, $k_3 = 0.5$, $k_4 = 0$, and initial value of the integrator $z(0) = 0$.

- An Output Feedback (OF) controller with discontinuous integral term, with controller gains $k_1 = 2\lambda^{\frac{2}{3}}$, $k_2 = 5\lambda^{\frac{1}{2}}$, $k_3 = 0.5\lambda$, $k_4 = 0$, $\lambda = 3$, observer gains $l_1 = 2L$, $l_2 = 1.1L^2$, $L = 4$, observer initial conditions $\hat{x}_1(0) = 0$, $\hat{x}_2(0) = 0$, and initial value of the integrator $z(0) = 0$.

- A Twisting controller, given by $u = -k_1 [x_1]^0 - k_2 [x_2]^0$, with gains $k_1 = 1.2$, $k_2 = 0.6$.

Perturbation $\rho(t) = 0.4 \sin(t)$
Simulations
Simulations

Lyapunov SMC

Jaime A. Moreno
UNAM
Simulations

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Simulations

![Control plot with multiple control methods: OF Control, SF Control, Twisting Control. The plot shows the control efforts over time, with each method shown in different colors and line styles.](image-url)
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Conclusions

- The Discontinuous Integral Controller
  - tracks exactly, in finite time and robustly
  - arbitrary references with bounded $\ddot{r}(t)$
  - despite arbitrary (time) Lipschitz
    perturbations/uncertainties, i.e. $\|\dot{\rho}(t)\| \leq \Delta$, $\Delta$ constant
  - without an Internal Model.
- Separate design of State Feedback and Observer;
- Neither continuous Observer nor continuous State Feedback Controller are insensitive to perturbations;
- Insensitivity against perturbations is achieved by discontinuous Integral Control;
- For implementation: $r(t)$ is required but not $\dot{r}(t)$ and $\ddot{r}(t)$.
- Design is Lyapunov-Based.
- Generalization to arbitrary order possible.
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Conclusions

1. Development of a Lyapunov based approach to HOSM and homogeneous control is an important task.

2. We require constructive methods to efficiently design controllers and observers for this class of systems.

3. We have provided some possible approaches. Each has its strengths and its weaknesses.

4. Still a lot of work has to be done

5. Other interesting approaches: Implicite Lyapunov Functions (ILF) by Lille Group!
Open Problems

2. Performance comparison of HOSM Controllers with other controllers (e.g. FOSM).
3. Is there a family of LF for HOSM providing necessary and sufficient stability conditions? Towards a more systematic Lyapunov Design.
4. (Truly) multivariable HOSM controllers and Observers. Some results for ST from Ch. Edwards, ...
5. Adaptive Algorithms. Important results from Y. Shtessel, F. Plestan, ...
6. Parameter estimation...
7. Implementation of HOSMs, Discretization methods,...
Thank you! Gracias!
A. Bacciotti and L. Rosier.

Liapunov functions and stability in control theory.

M. D. Choi, T. Y. Lam, and B. Reznick.

Sum of squares of real polynomials.

E. Cruz-Zavala and J. A. Moreno.

Improved convergence rate of discontinuous finite-time controllers.
In 19th IFAC World Congress (IFAC 2014), pages 8636–8641, Cape Town, South Africa, 2014.
E. Cruz-Zavala and J. A. Moreno.
A new class of fast finite-time discontinuous controllers.

E. Cruz-Zavala and J. A. Moreno.
Lyapunov approach to Higher-Order Sliding Mode design.

Emmanuel Cruz-Zavala and Jaime A. Moreno.
Homogeneous high order sliding mode design: A lyapunov approach.
S. Ding, A. Levant, and S. Li.

New families of high-order sliding-mode controllers.


S. Ding, A. Levant, and S. Li.

Simple homogeneous sliding-mode controller.


A. Levant.

Sliding order and sliding accuracy in Sliding Mode Control.

A. Levant.
Universal single-input single-output (siso) sliding-mode controllers with finite-time.


A. Levant.
Homogeneity approach to high-order sliding mode design.


J.A. Moreno and M. Osorio.
A Lyapunov approach to second-order sliding mode controllers and observers.

Jaime A. Moreno.


Y. Orlov.

Finite time stability and robust control synthesis of uncertain switched systems.

P. A. Parrilo.

*Structured Semidefinite Programs and Semialgebraic Geometry Methods in Robustness and Optimization*.
G. Pólya.
Über positive Darstellung von Polynomen.

A. Polyakov and A. Poznyak.
Reaching time estimation for “Super-Twisting” second-order sliding mode controller via Lyapunov function designing.

A. Polyakov and A. Poznyak.
Unified Lyapunov function for a finite-time stability analysis of relay second-order sliding mode control systems.
S. Prajna, A. Papachristodoulou, and P. A. Parrilo.


T. Sánchez and J.A. Moreno.

Construction of Lyapunov functions for a class of higher order sliding mode algorithms.

T. Sánchez and J.A. Moreno.
A constructive Lyapunov function design method for a class of homogeneous systems.

R. Santiesteban, L. Fridman, and J. A. Moreno.
Finite–time convergence analysis for “Twisting” controller via a strict Lyapunov function.
11th International Workshop on Variable Structure Systems (VSS), Mexico City, Mexico, June 2010.
T. Sánchez and J. A. Moreno.
Construction of lyapunov functions for High Order Sliding Modes.

V. Torres-Gonzalez, L.M. Fridman, and J.A. Moreno.
Continuous Twisting Algorithm.

VÃctor Torres-GonzÃ¡lez, Tonametl Sanchez, Leonid M. Fridman, and Jaime A. Moreno.
Design of continuous twisting algorithm.
N. Y. Zolotykh.

New modification of the Double Description Method for constructing the skeleton of a polyhedral cone.

*Computational Mathematics and Mathematical Physics, 52(1):146–156, 2012.*