IMPLICIT DISCRETIZATION IN SLIDING-MODE CONTROL: THEORY AND EXPERIMENTS

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1. Introduction on a simple first order example.
2. Extension to *n*th order LTI systems with matching perturbations.
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5. Extension to *n*th order LTI systems with parameter uncertainties.
6. Nonlinear systems.

The material that follows started with Vincent Acary, and collaborations with Yury Orlov, Olivier Huber, Andrey Polyakov, Franck Plestan, Ahcene Boubakir, Félix Miranda Villatoro, Fernando Castanos, Laurentiu Hetel.

First article on the subject: V. Acary, B. Brogliato, *Implicit Euler numerical scheme and chattering-free implementation of sliding mode systems*, Systems and Control Letters, vol.59, pp.284-293, 2010.

INTRODUCTION ON A SIMPLE SYSTEM

Let us consider the simple set-valued system:

$$\begin{cases} \dot{x}(t) = u(t) + d(x(t), t) \\ u(t) \in -\mathrm{sgn}(x(t)) \\ |d(x,t)| < 1 \text{ for all } x \text{ and } t. \\ \mathrm{sgn}(0) = [-1, 1]. \end{cases} \tag{1}$$

(a very simplified model of system with friction, or of sliding-mode controller with disturbance $d(x,t)$).

We can rely on Filippov or another mathematical framework (standard differential inclusions) to assert that:

1. The unique absolutely continuous solution (with suitable assumption on $d(t, x)$) converges in finite-time to the attractive "surface" $x = 0$.
2. The origin $x = 0$ is (finite-time) Lyapunov stable.
3. On the sliding surface one has $u = -d(t, x)$: exact compensation of the disturbance with *a priori* knowledge of only an upperbound of $d(t, x)$.

$\leadsto$ *The mathematical argument behind is the existence of a* **selection** *to the DI set-valued right-hand side (the controller u is the selection).*

Sliding-mode control is often presented as *discontinuous control* which can be approximated only by *infinite switching*: such an idea is fundamentally false ! The idea of a selection stems from the *continuity* of the **graph** of the multifunction sgn(·):

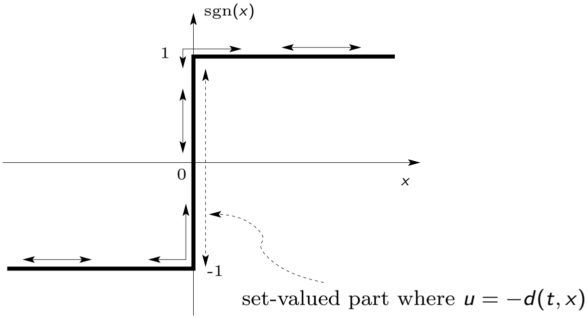


Figure: Continuity of the graph of the set-valued signum function.

1. As a consequence on the sliding surface the controller is independent of the control gain.
2. The set-valuedness of the controller is THE fundamental, crucial, indispensable property that makes SMC be what it is: robust and simple to tune.

   (the actuators have to allow for this: no step motor!)

The whole question is: how to realize (approximate) such set-valued behaviour in a discrete-time setting ?

and in a broader context: nth dimensional systems, mth dimensional attractive surface, parameter uncertainties, nonlinear systems.

Numerical analysis: discretization of (1) with an *implicit* Euler-like method (Moreau-Jean event-capturing scheme, 1987)

$$\begin{cases} x_{k+1} = x_k + h\, u_{k+1} + h\, d(x_k, t_k) \\ u_{k+1} \in -\mathrm{sgn}(x_{k+1}) \end{cases} \tag{2}$$

or more compactly:

$$\boxed{x_{k+1} \in x_k - h\, \mathrm{sgn}(x_{k+1}) + h\, d(x_k, t_k)} \tag{3}$$

$\rightsquigarrow$ this is a *generalized equation* for x_{k+1}, valid also at $x_{k+1} = 0$ (where sgn is set-valued)!

(a generalized equation is a nonlinear equation of the form

$$0 \in F(x)$$

where $F(x)$ is set-valued)

This generalized equation can also be written with u_{k+1} as the unknown by inverting the set-valued mapping:

$$\boxed{u_{k+1} \in -\mathrm{sgn}(x_{k+1}) \Longleftrightarrow x_{k+1} \in -N_{[-1,1]}(u_{k+1})}$$

where $N_{[-1,1]}(x)$ is the normal cone to the set $[-1,1]$ at x:

$$N_{[-1,1]}(x) = \begin{cases} \mathbb{R}_- & \text{if } x = -1 \\ \mathbb{R}_+ & \text{if } x = 1 \\ \{0\} & \text{if } |x| < 1 \\ \emptyset & \text{if } x \notin [-1,1] \end{cases} \tag{4}$$

We infer that:

$$\boxed{\frac{1}{h}x_k + u_{k+1} + d(x_k, t_k) \in N_{[-1,1]}(-u_{k+1})} \tag{5}$$

$\rightsquigarrow$ this is a *generalized equation* for u_{k+1}.

Due to their simplicity we can solve both generalized equations by inspection or graphically.

Whatever x_k and h and $d(x_k, t_k)$ there is always a unique solution in both x_{k+1} and u_{k+1}:

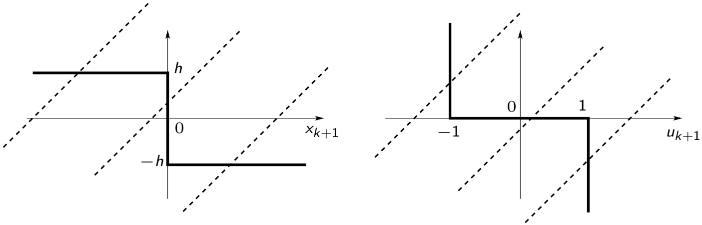


Figure: Graphical interpretation of the generalized equation.

Let us consider again:

$$\frac{1}{h}x_k + u_{k+1} + d(x_k, t_k) \in N_{[-1,1]}(-u_{k+1})$$

To analyse (6) let us make an aside on convex analysis. Let $K \subseteq \mathbb{R}^n$ be a non empty, convex and closed set. Let $x \in \mathbb{R}^n$, $y \in \mathbb{R}^n$ be two vectors, $M = M^T \in \mathbb{R}^{n\times n}$ a positive definite matrix. Then:

$$M(x - y) \in -N_K(x) \Leftrightarrow x = \mathrm{proj}_M(K; y)$$

$$\Longleftrightarrow x = \mathrm{argmin}_{z\in K}\tfrac{1}{2}(z - y)^T M(z - y)$$

We find that (6) is therefore *equivalent* to:

$$\boxed{u_{k+1} = -\mathrm{proj}\left([-1,1]; \frac{1}{h}x_k + d(x_k, t_k)\right)}$$

$\rightsquigarrow$ *The "controller" can be easily computed solving a quadratic programme (in higher dimensions).*

which allows us to advance the algorithm:

$$x_{k+1} = x_k + hd(x_k, t_k) - h\,\mathrm{proj}\left([-1,1]; \frac{1}{h}x_k + d(x_k, t_k)\right)$$

Let consider once again

$$\begin{cases} x_{k+1} = x_k + h\, u_{k+1} + h\, d(x_k, t_k) \\ \\ u_{k+1} \in -\mathrm{sgn}(x_{k+1}) \end{cases}$$

Direct inversion of $x_{k+1} \in x_k - h\mathrm{sgn}(x_{k+1}) + h\, d(x_k, t_k)$ yields:

$$\boxed{x_{k+1} \in (I + h\ \mathrm{sgn})^{-1}(x_k + hd_k)}$$

$\leadsto$ *Thus we have computed the operator $(I + h\ sgn)^{-1}$ which is single-valued (the $\in$ can be replaced by $=$).*

$\leadsto$ In the broader context of maximal monotone operators, the operator $J_A^{\mu}(x) = (I + \mu A)^{-1}(x)$ with $A(\cdot)$ maximal monotone, $\mu > 0$, is called the *resolvent* of A and is single valued, non expansive.

Behaviour on the sliding surface $\Sigma_{cont} = \{x | x = 0\}$, in discrete-time:

- **Invariance of the attractive surface:** It can be proved that $x_k = 0 \Rightarrow x_{k+1} = 0$: the discrete-time trajectory does not leave the attractive surface (which is invariant)

$$\Sigma_{dis} = \{x_k | x_k = 0\}.$$

Indeed using the closed-loop dynamics

$$x_{k+1} = x_k + hd(x_k, t_k) - h \operatorname{proj}\left([-1, 1]; \frac{1}{h}x_k + d(x_k, t_k)\right)$$

yields when $x_k = 0$:

$$\begin{aligned} x_{k+1} &= hd(0, t_k) - h \operatorname{proj}([-1, 1]; d(0, t_k)) \\ &= hd(0, t_k) - hd(0, t_k) = 0 \ \ (\text{since } |d(0, t_k)| < 1) \end{aligned}$$

- Thus there are no spurious numerical oscillations around Σ_{dis} (**no numerical chattering**). And we can give a rigorous meaning to a sliding surface in discrete time (no need for "quasi" sliding surface).
- Also on Σ_{dis} one has:

$$\begin{aligned} u_{k+1} &= -\mathrm{proj}\left([-1,1]; \tfrac{1}{h}x_k + d(0,t_k)\right) \\ &= -d(0,t_k) \end{aligned}$$

$\rightsquigarrow$ *The controller compensates for the disturbance (with one time-step delay with the choice for the discretization of $d(x,t)$).*

- **The attractive surface is attained in a finite number of steps.** Indeed let $\frac{1}{h}x_0 + d(x_0, t_0) > 1$, then the closed-loop dynamics is

$$x_1 = x_0 + hd(x_0, t_0) - h \underbrace{\operatorname{proj}\left([-1, 1]; \frac{1}{h}x_0 + d(x_0, t_0)\right)}_{=1}$$

$$= x_0 + hd(x_0, t_0) - h = x_0 - \delta_0 < x_0.$$

for some $\delta_0 > 0$. At the next step assume that still $\frac{1}{h}x_1 + d(x_1, t_1) > 1$ then $x_2 = x_1 - \delta_1 < x_1$ for some $\delta_1 > 0$ and so on. Due to $|d(t, x)| < 1$ we have that $\delta_0 > \delta$ and $\delta_1 > \delta$ for some $\delta > 0$. Thus $x_1 < x_0 - \delta$ and $x_2 < x_0 - 2\delta$. we can continue like this and we get $x_k < x_0 - k\delta$. Then $\frac{1}{h}x_k + d(x_k, t_k) < \frac{1}{h}x_0 - \frac{k}{h}\delta + d(x_k, t_k)$.

From the last inequality $\frac{1}{h}x_k + d(x_k, t_k) < \frac{1}{h}x_0 - \frac{k}{h}\delta + d(x_k, t_k)$ it becomes clear that for $k = k^*$ finite but large enough we obtain $\frac{1}{h}x_{k^*} + d(x_{k^*}, t_{k^*}) < 1$. Thus:

$$\begin{aligned} x_{k^*+1} &= x_{k^*} + hd(x_{k^*}, t_{k^*}) - h \underbrace{\mathrm{proj}\left([-1,1]; \frac{1}{h}x_{k^*} + d(x_{k^*}, t_{k^*})\right)}_{=\frac{1}{h}x_{k^*}+d(x_{k^*},t_{k^*})} \\ &= 0 \end{aligned}$$

$\rightsquigarrow$ *From the above item it follows that $x_k = 0$ for all $k > k^* + 1$.*

- We can prove (finite-time) Lyapunov stability of the fixed-point of the sliding variable dynamics: just take $V(x(k)) = x(k)^2$.
- **Controller gain insensitivity:** Changing $u_{k+1} \in -\mathrm{sgn}(x_{k+1})$ to $u_{k+1} \in -d_{\max}\,\mathrm{sgn}(x_{k+1})$ changes the reaching phase to Σ_{dis} but on the sliding phase on Σ_{dis} the values u_{k+1} remain unchanged. One way to see this is to consider the generalized equation for u_{k+1} in this case:

$$\frac{1}{h}x_k + u_{k+1} + d(x_k, t_k) \in \underbrace{N_{[-d_{\max}, d_{\max}]}(-u_{k+1})}_{=-N_{[-d_{\max}, d_{\max}]}(u_{k+1})}$$

so that equivalently

$$u_{k+1} = -\mathrm{proj}\left([-d_{\max}, d_{\max}]; \frac{1}{h}x_k + d(x_k, t_k)\right)$$

On the sliding surface we have $x_k = 0$ so that

$$\begin{aligned} u_{k+1} &= -\mathrm{proj}\left([-d_{\max}, d_{\max}]; d(x_k, t_k)\right) \\ &= -d(x_k, t_k) \end{aligned}$$

since $|d(x_k, t_k)| < d_{\max}$ by assumption.

$\rightsquigarrow$ *Thus increasing the gain $d_{\max}$ does not change u_{k+1}.*

Remark

In the undisturbed case $d(x, t) \equiv 0$, we obtain during the discrete-time sliding mode $u_k = 0$: no oscillations around the attractive surface neither in the ouput nor in the input (contrarily to the explicit discretization which has intrinsic oscillations, see [Galias et al]).

Graphical explanation:

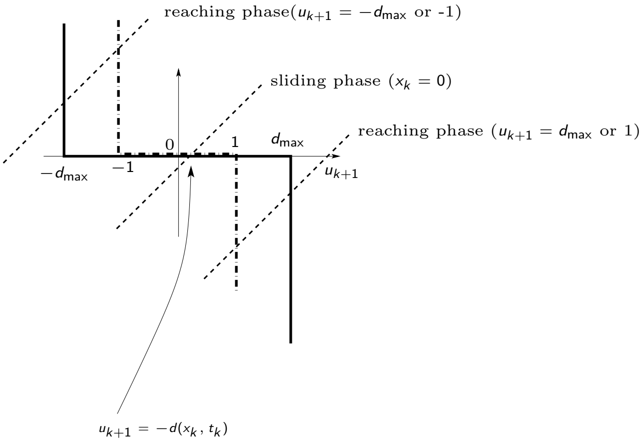


Figure: The controller does not change on Σ_{dis}.

The implicit discrete-time system behaves exactly like the continuous-time system according to Filippov's solutions:

1. Finite-time convergence, and trajectories stay on Σ_{cont} (decrease of system's dimension).
2. Lyapunov stability.
3. No digital chattering neither in the input nor in the output (sliding variable).
4. Control insensitive to gain on sliding surface, and compensates for the disturbance $u(t) = -d(x, t)$.
5. Same controller as in continuous-time.

$\rightsquigarrow$ *These properties are due to the set-valuedness of the closed-loop system.*

$\rightsquigarrow$ *In a sense, the implicit discretization allows to copy the continuous-time Filippov's differential inclusion behaviour, while keeping the input simple structure.*

We can use a **complementarity** framework to express and solve all of the above. Indeed the set-valued signum and its inverse (the normal cone to $[-1, 1]$) lend themselves to a complementarity description (as all piecewise-linear mappings, in fact).

$$x \in -\mathrm{sgn}(y) \Leftrightarrow y \in -N_{[-1,1]}(x)$$

and these two inclusions are in turn equivalent to:

$$\left\{ \begin{array}{l} x = \frac{\lambda_1 - \lambda_2}{2}, \quad \lambda_1 + \lambda_2 = 2 \\ 0 \leq \lambda_1 \perp y + |y| \geq 0 \\ 0 \leq \lambda_2 \perp y - |y| \geq 0 \end{array} \right. \Leftrightarrow \left\{ \begin{array}{l} y = \lambda_1 - \lambda_2 \\ u_1 = 1 + x, \; u_2 = 1 - x \\ 0 \leq u_1 \perp \lambda_1 \geq 0 \\ 0 \leq u_2 \perp \lambda_2 \geq 0. \end{array} \right. \tag{6}$$

Let us check the second representation in (6) by inspection:

- Let $|x| < 1$: then $u_1 > 0$ and $u_2 > 0$ so that $\lambda_1 = \lambda_2 = 0$ so that $y = 0$.
- Let $x = -1$: then $u_1 = 0 \Rightarrow \lambda_1 \geq 0$, and $u_2 = 2 \Rightarrow \lambda_2 = 0$, so that $y = \lambda_1 \geq 0$.
- Let $x = 1$: then $u_2 = 0 \Rightarrow \lambda_2 \geq 0$, and $u_1 = 2 \Rightarrow \lambda_1 = 0$, so that $y = -\lambda_2 \leq 0$.

We indeed recover the normal cone to $[-1, 1]$.

Same could be done for the first representation in (6).

Let us now analyse our discrete-time system within the complementarity approach. Using the above material we can write the closed-loop system equivalently as:

$$\begin{cases} (a) \quad u_{k+1} = \frac{\lambda^1_{k+1} - \lambda^2_{k+1}}{h} - \left(\frac{x_k}{h} + d_k\right) \\ \\ (b) \quad 0 \leq M\lambda_{k+1} + q_k \perp \lambda_{k+1} \geq 0 \end{cases} \tag{7}$$

with $M = \begin{pmatrix} \frac{1}{h} & -\frac{1}{h} \\ -\frac{1}{h} & \frac{1}{h} \end{pmatrix}$, $q_k = \begin{pmatrix} 1 - \left(\frac{x_k}{h} + d_k\right) \\ 1 + \left(\frac{x_k}{h} + d_k\right) \end{pmatrix}$,
$\lambda_k = \begin{pmatrix} \lambda^1_k \\ \lambda^2_k \end{pmatrix}$.

$\rightsquigarrow$ *The problem in (7) (b) is a Linear Complementarity Problem (LCP) with unknown the multiplier* λ_{k+1}.

An aside on LCPs:

Let $M \in \mathbb{R}^{n \times n}$, $q \in \mathbb{R}^n$. An LCP is a nonsmooth problem with unknown $\lambda \in \mathbb{R}^n$, of the form:

$$\boxed{M\lambda + q \geq 0, \quad \lambda \geq 0, \quad \lambda^T(M\lambda + q) = 0} \tag{8}$$

which we rewrite compactly as $0 \leq \lambda \perp M\lambda + q \geq 0$.

Existence and uniqueness of solutions to LCPs is a widely studied subject. Central result of complementarity theory:

Theorem

Let M be a P-matrix, then the LCP has a unique solution λ^* for any q, and vice-versa.

$\rightsquigarrow$ *Many extensions for non P-matrices (semi positive definite, co-positive, P_0 matrices* etc*).*

In our case $M = M^T$ is positive semi definite. But it can be proved that the LCP in (7) (b) is always *feasible*, that is there always exist λ_{k+1} such that $\lambda^1_{k+1} \geq 0$, $\lambda^2_{k+1} \geq 0$, $M\lambda_{k+1} + q_k \geq 0$.

Since M is positive semi definite, feasibility implies *solvability*: our LCP(λ_{k+1}) always has a solution whatever q_k.

Moreover due again to M being positive semi definite, two solutions β_{k+1} and ζ_{k+1} of LCP(λ_{k+1}) always satisfy $\zeta^1_{k+1} - \zeta^2_{k+1} = \beta^1_{k+1} - \beta^2_{k+1}$.

$\Longrightarrow$ We infer that $u_{k+1} = \frac{\lambda^1_{k+1} - \lambda^2_{k+1}}{h} - \left(\frac{x_k}{h} + d_k\right)$ is uniquely defined for any $\frac{x_k}{h} + d_k$.

We conclude that we can calculate u_{k+1} by solving the LCP in (7) (b) with a suitable numerical LCP solver: just another way to compute the projection in (6) !

We can embedd our system into the class of differential inclusions with maximal monotone right-hand side:

$$\dot{x}(t) + f(x(t), t) \in -A(x(t)) \tag{9}$$

where $A : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is set-valued maximal monotone, *i.e.*: for all $x_1 \in \mathrm{dom}(A)$, $x_2 \in \mathrm{dom}(A)$, $y_1 \in A(x_1)$, $y_2 \in A(x_2)$, we have:

$$\langle x_1 - x_2, y_1 - y_2 \rangle \geq 0$$

and the graph cannot be extended without destroying the monotonicity (*simple example: signum multifunction*).

Existence and uniqueness of Lipschitz solutions is then a classical result. Implicit Euler discretization reads as:

$$\begin{cases} x_{k+1} = x_k - hf(x_k, t_k) - h\lambda_{k+1} \\ \lambda_{k+1} \in A(x_{k+1}) \end{cases} \tag{10}$$

Results of convergence, order [Bastien, Schatzman]. However not sufficient for us !

Let us now briefly have a look at a "natural" discretization widely used in sliding-mode control applications: *the explicit Euler method*:

$$\begin{cases} x_{k+1} = x_k + h\,u_k + h\,d(x_k, t_k) \\ u_k \in \underbrace{-\mathrm{sgn}(x_k)}_{\text{here } x_k \text{ instead of } x_{k+1}!} \end{cases} \tag{11}$$

Generalized equation to calculate x_{k+1} and u_k at $x_k = 0$:

$$\boxed{\begin{array}{l} x_{k+1} = h\,u_k + h\,d(0, t_k) \\ u_k \in -\mathrm{sgn}(0) \end{array}} \tag{12}$$

$\rightsquigarrow$ *No rule to select a value inside* $[-1, 1]$.

$\rightsquigarrow$ *Implies oscillations around* Σ_{dis} *with* $u_k = \pm 1$ *(*$\approx$ *step motor!): numerical chattering of the input and of the output.*

What happens with an explicit controller $u_k = -\operatorname{sgn}(x_k)$, which yields the closed-loop dynamics:

$$x_{k+1} = x_k - h \operatorname{sgn}(x_k) + h\, d(x_k, t_k)$$

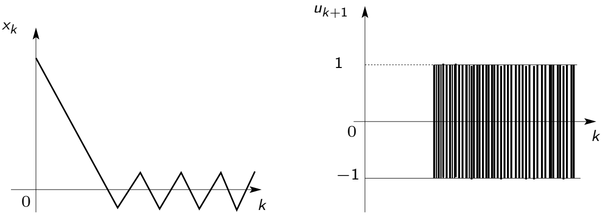


Figure: SMC with explicit Euler discretization.

More can be found in the results by Galias et al: existence of limit cycles (numerical chattering) for SMC systems.

Remark

It should be stressed that chattering in both input and output exists with the explicit Euler discretization, even without perturbations ! The chattering is intrinsic to the method.

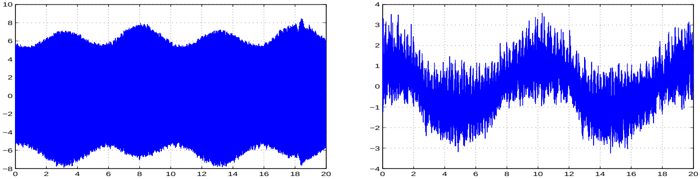
The input is a bang-bang controller whose magnitude increases with the controller gain, even during the sliding-mode phase (which in fact does not exist!!). It has no chance to converge to the continuous-time set-valued input!

Moreover the explicit method can yield instability in nonlinear homogeneous systems [Levant 2013, Efimov et al 2017].

Usual trick is the regularization of signum set-valued function. However :

1. Tuning of regularized signum parameters (saturation width) and h to alleviate the chattering is not always simple (even in simplest cases !).
2. Case of several sliding surfaces: unclear what happens in the neighborhood of co-dimension ≥ 2 attractive surfaces.
3. Loss of accuracy (think of control), loss of sticking modes (think of dry friction).
4. Adds parameters not present in the continuous-time framework.

To give a rough idea, a typical discrepancy between implicit and explicit methods obtained from experiments:



$\rightsquigarrow$ *The implicit discretization allows to drastically decrease the controller magnitude and the chattering.*

$\rightsquigarrow$ *The implicit controller reflects the perturbation.*

The above was more numerical analysis (known "perturbation"), let us now deal with its **sliding-mode control implementation**.

In sliding-mode control, the perturbation $d(x,t)$ is unknown (only an upperbound is supposed to be known), so the above generalized equation to compute the control input u_{k+1} is no longer valid. We proceed as follows:

$$\begin{cases} x_{k+1} = x_k + h\, u_{k+1} + h\, d(x_k, t_k) & \text{(plant dynamics)} \\ \boxed{\begin{array}{l} \tilde{x}_{k+1} = x_k + h\, u_{k+1} \\ u_{k+1} \in -\mathrm{sgn}(\tilde{x}_{k+1}) \end{array}} & \text{(virtual system)} \end{cases}$$

In the LTI case, the plant is discretized with exact ZOH method. Otherwise use an approximate Euler discretization.

The virtual (undisturbed) system yields the generalized equation:

$$\boxed{\begin{array}{l} \tilde{x}_{k+1} = x_k + h\ u_{k+1} \\ \\ u_{k+1} \in -\mathrm{sgn}(\tilde{x}_{k+1}) \end{array}}$$

which we can treat as above to compute the controller u_{k+1}. First we invert the set-valued relation as:

$$u_{k+1} \in -\mathrm{sgn}(\tilde{x}_{k+1}) \Leftrightarrow \tilde{x}_{k+1} \in N_{[-1,1]}(-u_{k+1})$$

Thus we obtain from (13) $x_k + hu_{k+1} \in N_{[-1,1]}(-u_{k+1})$ which is equivalent to:

$$\boxed{u_{k+1} = -\mathrm{proj}\left([-1,1]; \frac{x_k}{h}\right)}$$

The implicit controller is non anticipative. We obtain the closed-loop system:

$$\boxed{x_{k+1} = x_k + h\, d(x_k, t_k) - h \text{ proj}\left([-1,1]; \frac{x_k}{h}\right)}$$

It differs from the foregoing one which was:

$$x_{k+1} = x_k + hd(x_k, t_k) - h \text{ proj}\left([-1,1]; \frac{1}{h}x_k + d(x_k, t_k)\right)$$

$\rightsquigarrow$ *This time we have no chance to get $x_k = 0$, rather the discrete-time sliding surface will be defined from $\tilde{x}_k = 0$.*

$\rightsquigarrow$ *When there is no disturbance then $\tilde{x}_k = x_k$.*

Reaching phase:

1. **Case $x_k > h$:** from the closed-loop dynamics $x_{k+1} = x_k + h\, d(x_k, t_k) - h \operatorname{proj}\left([-1, 1]; \frac{x_k}{h}\right)$ it follows that

$$x_{k+1} = x_k - h \underbrace{(1 - d(x_k, t_k))}_{>0}$$

so $x_{k+1} = x_k - h\,\delta$ for some $\delta > 0$.

$\rightsquigarrow$ *Strict decrease of the sliding variable at each step.*

2. **Case $x_k < -h$:** similar calculations yield $x_{k+1} = x_k + h\,\delta$:

$\rightsquigarrow$ *Strict increase of the sliding variable at each step.*

3. **Conclusion:** Starting from $|x_0| > h$, after a finite number of m steps, we enter the layer $\Sigma_h = \{x_k \mid |x_k| < h\}$.

Analysis inside Σ_h:

1. **Case** $|x_k| < h$**:** from the closed-loop dynamics $x_{k+1} = x_k + h\,d(x_k, t_k) - h \operatorname{proj}\left([-1, 1]; \frac{x_k}{h}\right)$ it follows that

$$x_{k+1} = hd(x_k, t_k) < h$$

since $d(x_k, t_k) < 1$, so that $x_{k+1} < h$ and we stay in this situation for all future steps.

$\rightsquigarrow$ *The disturbance is attenuated by a factor h.*

2. Moreover in Σ_h, we have by direct calculation that $\tilde{x}_k = 0$: the system is in the discrete-time sliding mode.
3. In the discrete-time sliding mode, $u_{k+1} = -d(x_k, t_k)$: the controller compensates for the disturbance with a delay equal to h.

Numerical analysis *vs.* discrete-time SMC

- Similar discretizations, but different objectives and different set-valued parts in general.
- Experimental validations in both cases:
  - Numerical simulations: mechanical systems with set-valued frictional contact (unilateral + bilateral constraints)
  - SMC: Pneumatic and electro-mechanical systems.
- In both cases, the explicit method (replace $\mathrm{sgn}(x_{k+1})$ by $\mathrm{sgn}(x_k)$, or $\mathrm{sgn}(\tilde{x}_{k+1})$ by $\mathrm{sgn}(x_k)$):
  - produces spurious oscillations around the attractive surface (chattering) and in the input
  - is unable to guarantee Lyapunov stability and finite-time convergence.
  - produces a control input that grows when control gains grows.

Overview of the obtained results:

- LTI systems, Lagrangian systems, scalar nonlinear systems, homogenous systems, fractional order systems.
- 1st order SMC, maximal monotone controllers, twisting and super-twisting algorithms
- robustness w.r.t. matched, unmatched perturbations, and parametric uncertainties
- Lyapunov stability (global, semi-global), finite-time convergence to the attractive surface, chattering suppression (in both input and output), input convergence (in the variation and in the infinity norm), clear notion of discrete-time sliding-mode with $\tilde{\sigma}_k = \tilde{x}_k = 0$ (but not $\sigma_k = x_k$).

1st ORDER SMC CASE FOR LTI SYSTEMS

Let us now deal with systems of arbitrary dimension and matched disturbances:

$$\boxed{\dot{x}(t) = Ax(t) + Bu(t) + B\varphi(t)}$$

with $||\varphi(t)||_1 = \sum_{i=1}^{m} |\varphi_i(t)| \leq m\varphi_{\max}$ and $|\varphi^i|_\infty \leq \varphi_{\max}$.

The sliding surface is chosen classically as $\Sigma = \{x \mid Cx = 0\}$, $C \in \mathbb{R}^{m\times n}$, $CB \in \mathbb{R}^{m\times m}$ full rank.

We define the first-order SMC as $u = u^{eq} + u^s$, with $u^{eq} = -(CB)^{-1}CAx$. The sliding variable dynamics is then given by the differential inclusion:

$$\begin{cases} \dot{\sigma}(t) = CB\ u^s(t) \\ u^s(t) \in -\alpha\ \mathrm{Sgn}(\sigma(t)) \end{cases}$$

where $\mathrm{Sgn}(\sigma) = (\mathrm{sgn}(\sigma_1), \mathrm{sgn}(\sigma_2), \ldots, \mathrm{sgn}(\sigma_m))^T$.

The (approximate) Euler discretization of the plant is given by:

$$x_{k+1} = x_k + hAx_k + hBu_{k+1} + hB\varphi_{k+1}$$

Remark

We can also use the exact discretization ZOH method:

$$x_{k+1} = e^{Ah}x_k + B^*\bar{u}_k^{eq} + B^*\bar{u}_k^s + p_k$$

with $B^* = \int_{t_k}^{t_{k+1}} e^{A(t_{k+1}-\tau)}Bd\tau$, $p_k = \int_{t_k}^{t_{k+1}} e^{A(t_{k+1}-\tau)}B\varphi(\tau)d\tau$,

and the controllers $\bar{u}^{eq}$ *and* $\bar{u}^s$ *are sampled control laws defined as*

$$\bar{u}^{eq}(t) = u_k^{eq}, \quad \bar{u}^s(t) = u_k^s$$

for all $t \in [t_k, t_{k+1})$.

Let us use the Euler discretization. The nominal system is defined as:

$$\tilde{x}_{k+1} = (I + hA)x_k + hBu_{k+1}$$

and the implicit Euler controller is defined as:

$$\boxed{\begin{array}{l} u_{k+1} = -(CB)^{-1}CAx_k - \alpha(CB)^{-1}\tau_{k+1} \\ \\ \tau_{k+1} \in \mathrm{Sgn}(C\tilde{x}_{k+1}) \end{array}}$$

with $\alpha > m||CB||\varphi_{\max}$.

Let $\sigma_k = Cx_k$ and $\tilde{\sigma}_k = C\tilde{x}_k$. The closed-loop system reads as:

$$\begin{cases} \boxed{\begin{array}{l} \tilde{\sigma}_{k+1} = \sigma_k - \alpha h\, \tau_{k+1} \\ \\ \tau_{k+1} \in \mathrm{Sgn}(\tilde{\sigma}_{k+1}) \end{array}} \quad \text{(virtual system)} \\ \\ \underbrace{\sigma_{k+1} = \sigma_k - \alpha h\, \tau_{k+1} + hCB\varphi_{k+1}}_{\text{System's dynamics}} \end{cases}$$

Proceeding as above we calculate the (non anticipative) controller as:

$$\boxed{\tau_{k+1} = \mathrm{proj}\left([-1, 1]^m; \frac{1}{\alpha h} Cx_k \right)}$$

The complete control law is therefore given at $t = t_k$ by:

$$u_{k+1} = \underbrace{-(CB)^{-1}CAx_k}_{\approx\text{continuous feedback}} \quad -\alpha(CB)^{-1}\underbrace{\text{proj}\left([-1,1]^m; \frac{1}{\alpha h}Cx_k\right)}_{\approx\text{set-valued feedback}} \tag{13}$$

The following properties hold:

1. The discrete-time sliding surface $\tilde{\sigma}_k = 0$ is attained after a finite number of steps for any bounded initial data.
2. The disturbance is attenuated by a factor h on during the discrete sliding mode.
3. During the sliding mode, the controller compensates for the disturbance with a delay h.
4. During the sliding mode the set-valued controller magnitude is independent of the gain α.
5. Finite-time Lyapunov stability holds (for the sliding variable dynamics).

Let us use the ZOH discretization. We obtain nice convergence results for the input. Let us recall the discrete-time dynamics:

$$x_{k+1} = e^{Ah} x_k + B^* \bar{u}_k^{eq} + B^* \bar{u}_k^s + p_k$$

with the controllers $\bar{u}^{eq}$ and $\bar{u}^s$ are sampled control laws defined as

$$\bar{u}^{eq}(t) = u_k^{eq}, \quad \bar{u}^s(t) = u_k^s$$

for all $t \in [t_k, t_{k+1})$. We define $\sigma_k = Cx_k$ and $\tilde{\sigma}_k$ such that:

$$\boxed{\begin{array}{l} \tilde{\sigma}_k = \sigma_k + CB^* \; \bar{u}_k^s \\ \\ \bar{u}_k^s \in -\alpha \, \mathrm{Sgn}(\tilde{\sigma}_k) \end{array}} \tag{14}$$

which is the generalized equation to compute the controller.

Remark

Even in the ideal unperturbed case, in general $\tilde{\sigma}_k \neq \sigma_k$, because errors are introduced by the discretization of the "equivalent" continuous controller u^{eq}. Only if an exact discrete equivalent controller is used, are they equal.

However if we apply the exact ("equivalent" part of u) controller:

$$\bar{u}_k^{eq} = (CB^*)^{-1} C(I - e^{Ah})x_k \tag{15}$$

then in the undisturbed case $\tilde{\sigma}_k = \sigma_k$.

The discrete-time sliding mode

Definition (Discrete-time sliding mode)

When $\bar{u}_k^s$ is in the interior of $[-\alpha, \alpha]^p$, we say that the closed-loop system is in the *discrete-time sliding phase*. The inclusion $-\widetilde{\sigma}_{k+1} \in \mathcal{N}_{[-\alpha,\alpha]^p}(\bar{u}_k^s)$ implies that in this case the normal cone is reduced to the singleton $\{0\}$. Thus the sliding variable $\widetilde{\sigma}_k$ is zero in the discrete-time sliding phase.

Computation of the controller: Using the same manipulations as above we find that it is the solution of the generalized equation:

$$0 \in \sigma_k + CB^* \ \bar{u}_k^s + N_{[-\alpha,\alpha]^m}(\bar{u}_k^s) \tag{16}$$

(i) Does (16) have a solution ? **(ii)** If yes is it unique ? **(iii)** How to calculate it online ? **(iv)** Does $\bar{u}^s(\cdot)$ converge to $u^s(\cdot)$?

Remark

Application of the ZOH method requires the knowledge of A and B. Relaxing exact knowledge of A (replace it by $A + \Delta A$) is possible but with more complex design and analysis.

Here we want to avoid computing B^* at each time step, so we keep it in (16).

To answer the questions **(i)–(iii)** we first transform (16) into the Affine Variational Inequality (AVI):

$$\boxed{\begin{array}{l}\text{Find } z \in [-\alpha, \alpha]^m \text{ such that:} \\ \\ (y - z)^T(\sigma_k + CB^*\, z) \geq 0, \ \text{ for all } y \in [-\alpha, \alpha]^m.\end{array}} \tag{17}$$

(the equivalence between (16) and (17) follows directly from the definition of the normal cone to a convex set) (proofs in [Huber, Acary, Brogliato, TACON November 2016])

Lemma (controller existence)

The AVI (17) always has a solution.

Lemma (controller uniqueness)

The AVI (17) has a unique solution for all σ_k if and only if CB^* is a P-matrix.

(computation can be performed numerically with a suitable algorithm for LCPs or QPs)

Lemma

Suppose that CB is positive definite. There exists an interval $\mathcal{I} = (0, h^*] \subset \mathbb{R}_+$, $h^* > 0$, such that if the sampling period $h \in \mathcal{I}$, then CB^* is positive definite.

Proposition (Finite-time convergence)

Let CB^* be positive definite. Let also α be such that $||Cp_k|| < \alpha\beta$, where β is the smallest eigenvalue of $\frac{1}{2}(CB^* + (CB^*)^T)$. Then the perturbed closed loop system:

$$\begin{cases} \tilde{\sigma}_k = \sigma_k + CB^* \ \bar{u}_k^s \\ \bar{u}_k^s \in -\alpha \ Sgn(\tilde{\sigma}_k) \\ \sigma_{k+1} = \sigma_k + CB^* \ \bar{u}_k^s + Cp_k \end{cases} \tag{18}$$

enters the discrete time sliding phase in finite time and stays in it with $\sigma_{k+1} = Cp_k$ and $\bar{u}_k^s = -(CB^*)^{-1}Cp_k$. Furthermore if $h \in (0, h^*]$ then there exists an upperbound T^* on the duration of the reaching phase.

About the condition $||Cp_k|| < \alpha\beta$:

If $h > 0$ is small enough then $(||Cp_k|| < \alpha\beta) \implies (||\varphi||_{\infty,\mathbb{R}_+} < \alpha)$. Equivalence holds if $m = 1$ (co-dimension one sliding surface).

The proof is led with the Lyapunov function $V(\sigma_k) = \alpha||\sigma_k||_1$, showing in passing finite-time global **Lyapunov stability**.

Remark

This means that we completely depart from usual criteria in the discrete-time SMC literature like $(\sigma_{k+1} - \sigma_k)\sigma_k < 0$. Here we rely on the system's parameters (CB^* positive definite) to guarantee the finite-time stability.

Let us analyse the convergence of $\bar{u}$ to u during the discrete-time sliding phase (*i.e.*, after $T^* < \infty$).
Consider once again the discrete-time closed loop system

$$\begin{cases} \tilde{\sigma}_k = \sigma_k + CB^* \, \bar{u}_k^s \\ \bar{u}_k^s \in -\alpha \, \mathrm{Sgn}(\tilde{\sigma}_k) \\ \sigma_{k+1} = \sigma_k + CB^* \, \bar{u}_k^s + Cp_k \end{cases} \tag{19}$$

Proposition (Input convergence, infinity norm)

Consider the discrete-time closed-loop system given by (19). Let $\{h_n\}_{n\in\mathbb{N}}$ *be any strictly decreasing sequence of positive numbers converging to* 0 *and with* $h_0 < h^*$*. Suppose that the perturbation* $\varphi : \mathbb{R} \to \mathbb{R}^p$ *is uniformly continuous, that* CB *is positive definite and that* $\alpha > 0$ *is chosen such that* $||Cp_k|| < \alpha\beta$ *for each sampling period* h_n*. Then for any interval* $S \subseteq [T^*, \infty)$*,*

$$\lim_{h_n \to 0} \|\bar{u}^s - u^s\|_{\infty, S} = 0.$$

We have also a result of convergence of the variation of the control signal:

Definition (Variation of a function)

Let $f : \mathbb{R} \to \mathbb{R}^m$ be a right-continuous step function, discontinuous at finitely many time instants t_k and $t_0, T \in \mathbb{R}$ with $t_0 < T$. The variation of f on $[t_0, T]$ is defined as:

$$\mathrm{Var}_{t_0}^{T}(f) \triangleq \sum_{k} \|f(t_k) - f(t_{k-1})\|, \tag{20}$$

with $k \in \mathrm{N}^*$ such that $t_k \in (t_0, T]$. If f is continuously differentiable with bounded derivatives then the variation of f on $[t_0, T]$ is defined as:

$$\mathrm{Var}_{t_0}^{T}(f) \triangleq \int_{t_0}^{T} \|\dot{f}(\tau)\| \mathrm{d}\tau. \tag{21}$$

Proposition (Convergence in variation)

Suppose that CB is positive definite, and φ is a real-valued continuously differentiable with bounded derivative function. Let $\{h_n\}_{n\in\mathrm{N}}$ be any strictly decreasing sequence of positive numbers converging to 0 with $h_0 < h^*$. Let α be chosen such that $||Cp_k|| < \alpha\beta$ is satisfied for each h_n. Let $T > T^*$ with T^* defined in Proposition 1. Then

$$\lim_{h_n\to 0} \mathrm{Var}_{T^*}^{T}(\bar{u}^s) = \mathrm{Var}_{T^*}^{T}(u^s).$$

And we finally have the following:

Corollary (Controller gain insensitivity)

Suppose that α *is such that for all* $k \in \mathbb{N}$, $\|Cp_k\| \leq \alpha\beta$. *Then even if the controller gain is increased to* $\alpha' > \alpha$, *the control input* $\bar{u}^s$ *does not change in the discrete-time sliding phase.*

Proof.

From the proof of Proposition 1, we have that $\bar{u}_k^s$ is uniquely defined as the solution to

$$\begin{cases} \tilde{\sigma}_k = \sigma_k + CB^* \ \bar{u}_k^s \\ \bar{u}_k^s \in -\alpha \ \mathrm{Sgn}(\tilde{\sigma}_k) \end{cases} \tag{22}$$

and is equal to $-(CB^*)^{-1}Cp_k$ which does not depend on the controller gain. □

With the exact ZOH discretization we obtain the following properties when the implicit method is used:

1. The controller is easily calculated at each time step (projection, or solving an AVI).
2. The controller has powerful convergence properties towards its set-valued continuous-time counterpart.
3. The controller keeps the simple structure of its set-valued continuous-time counterpart.
4. Numerical chattering is (in theory and numerically) avoided in both the input and the output.
5. The sliding mode is well-defined in discrete-time and the controller is gain-insensitive inside the sliding mode.
6. The sliding-mode is attained in a finite number of steps and Lyapunov global stability holds.
7. The method applies to co-dimension $m \geq 2$ attractive surfaces.

EXPERIMENTAL VALIDATIONS

Results have been obtained:

- on a electropneumatic system at LS2N Nantes,
- on an inverted pendulum at Laboratory CRIStAL in Lille.
- Experiments led mainly by Olivier Huber in his Ph.D. thesis (with the help of F. Plestan, A. Boubakir, L. Hetel, B. Wang).

Figure: Photography of the electropneumatic system (LS2N, Ecole Centrale de Nantes, France).

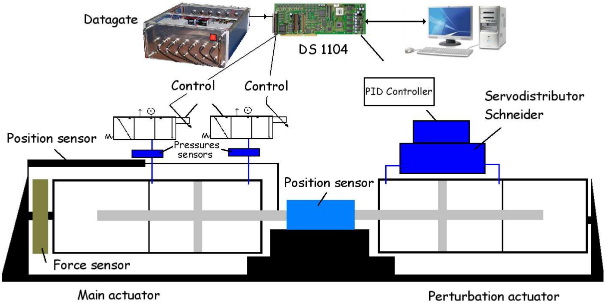


Figure: Scheme of the electropneumatic system (LS2N, Ecole Centrale de Nantes, France).

The goal is to control the "main actuator" motion, while the "perturbation actuator" produces an external disturbance force. Both actuators are controlled by servodistributors.

The model is divided in two parts: two first equations concern the pressure dynamics in each chamber whereas the motion of the actuator is described by the two last equations:

$$\left\{\begin{array}{lcl} \dot{p}_P & = & \frac{krT}{V_P(y)}[\varphi_P + \psi_P\, u - \frac{S}{rT} p_P v] \\ \dot{p}_N & = & \frac{krT}{V_N(y)}[\varphi_N - \psi_N\, u + \frac{S}{rT} p_N v] \\ \dot{v} & = & \frac{1}{M}[S\,(p_P - p_N) - b_v v - F] \\ \dot{y} & = & v, \end{array}\right. \tag{23}$$

with p_P (reps. p_N) the pressure in the P (resp. N) chamber, y and v being the position and velocity of the actuator. The force F is the disturbance.

Thus we can write the dynamics as $\dot{x} = f(x) + g(x)u$ with $f(\cdot)$ and $g(\cdot)$ defined as

$$f(x) = \begin{bmatrix} \frac{krT}{V_P(y)}\left[\varphi_P - \frac{S}{rT} p_P v\right] \\ \frac{krT}{V_N(y)}\left[\varphi_N + \frac{S}{rT} p_N v\right] \\ \frac{1}{M}\left[S\left(p_P - p_N\right) - b_v v - F\right] \\ v \end{bmatrix}, \quad g(x) = \begin{bmatrix} \frac{krT}{V_P(y)} \psi_P \\ -\frac{krT}{V_N(y)} \psi_N \\ 0 \\ 0 \end{bmatrix},$$

The sliding variable is defined as:

$$\sigma(x,t) = \ddot{e} + \lambda_1 \dot{e} + \lambda_0 e \tag{24}$$

with $e \stackrel{\Delta}{=} y - y_d(t)$, $y_d(t)$ being the desired trajectory, supposed to be sufficiently differentiable. The coefficients $\lambda_1,\ \lambda_0$ are defined such that the polynomial $Q(z) = z^2 + \lambda_1 z + \lambda_0$ is Hurwitz.

After some manipulations one obtains:

$$\begin{aligned} \dot{\sigma} &= \Psi(x,t) + \Phi(x)u \\ &= \Psi_n(x,t) + \Delta\Psi(t) + [\Phi_n(x) + \Delta\Phi(t)]\ u \end{aligned}$$

such that Ψ_n, Φ_n are the nominal functions and $\Delta\Psi, \Delta\Phi$ are the uncertain terms.

Let us consider the control law:

$$\boxed{u = \frac{1}{\Phi_n}\left[-\Psi_n + v\right].}$$

By applying (25) in (25), one gets

$$\boxed{\dot{\sigma} = \frac{\Delta\Phi}{\Phi_n}\Psi_n + \Delta\Psi + \left[1 + \frac{\Delta\Phi}{\Phi_n}\right] v.}$$

The controller v is a set-valued input defined as

$$\boxed{v \in -G\mathrm{sgn}(\sigma)}$$

with G tuned sufficiently large.

Some facts and assumptions:

1. The functions Ψ and Φ are bounded in the physical working domain (so the uncertain terms are also bounded).
2. $\Delta\Phi$ is sufficiently small with respect to Φ_n to ensure that $1 + \frac{\Delta\Phi}{\Phi_n} > 0$. From a practical point of view, this assumption is not too strong: it simply means that the uncertainties are small compared to the nominal values.
3. The gain has to be tuned as $G > \dfrac{\max\left|\dfrac{\Delta\Phi}{\Phi_n}\Psi_n + \Delta\Psi\right| + \eta}{\min\left[1 + \dfrac{\Delta\Phi}{\Phi_n}\right]}$. It can be shown that, over the trajectories and in the working domain, the term $\dfrac{\Delta\Phi}{\Phi_n}\Psi_n + \Delta\Psi$ is bounded whenever $1 + \dfrac{\Delta\Phi}{\Phi_n} > 0$.

Accepting all these assumptions which more or less recast the electropneumatic system into our theoretical framework, we have to control the system:

$$\boxed{\begin{array}{l} \dot{\sigma} = \dfrac{\Delta\Phi}{\Phi_n}\Psi_n + \Delta\Psi + \left[1 + \dfrac{\Delta\Phi}{\Phi_n}\right] v. \\[2em] v \in -G\,\mathrm{sgn}(\sigma) \end{array}}$$

with three different controllers:

- **Explicit sliding mode control** (with sgn(·) function)

$$v_k = -G\mathrm{sgn}(\sigma_k),$$

- **Explicit saturated sliding mode control** (with sat(·) function)

$$v_k = -G\mathrm{sat}(\sigma_k, \epsilon),$$

with

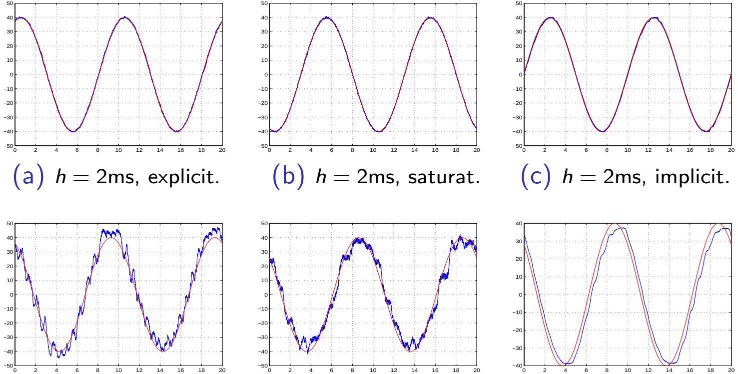
$$\mathrm{sat}(\sigma_k, \epsilon) = \begin{cases} \mathrm{sgn}(\sigma_k) \text{ if } |\sigma_k| \geq \epsilon \\ \sigma_k \text{ if } |\sigma_k| < \epsilon. \end{cases}$$

- **Implicit sliding mode control** (with sgn(·) multifunction)

$$v_k \in -G\mathrm{sgn}(\sigma_{k+1})$$

(implemented with a projection).

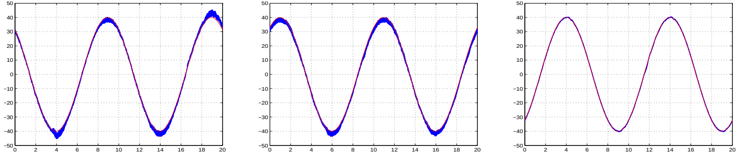
- The controllers have been implemented with three feedback gains $G = 10^4$, $G = 10^5$, $G = 10^6$ and five sampling times 1 ms, 2 ms, 5 ms, 10 ms and 15 ms.
- The length of the interval of study is 20 seconds.
- The saturation input has been tested for six different values of the saturation width, with $h = 1$ ms, and the unitless width $\epsilon = 0.1$ (the other widths which have been tested yielded similar results).
- The comparisons are mainly made with respect to:
  1. the magnitude and chattering of the inputs u and v,
  2. the tracking error e.



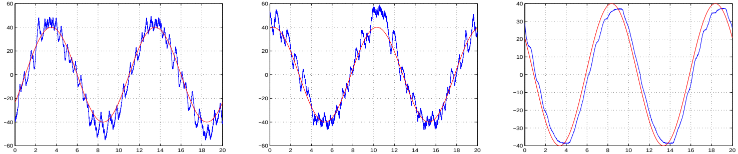
(a) $h = 2$ms, explicit. (b) $h = 2$ms, saturat. (c) $h = 2$ms, implicit.

(d) $h = 15$ms; explicit. (e) $h = 15$ms, saturat. (f) $h = 15$ms, implicit.

Figure: Real position y (mm) in blue and y_d (mm) in red, under $h = 2$ms and $h = 15$ms for $\mathbf{G} = 10^4$.



(a) $h = 2$ms, explicit. (b) $h = 2$ms, saturat. (c) $h = 2$ms, implicit.



(d) $h = 15$ms, explicit. (e) $h = 15$ms, saturat. (f) $h = 15$ms, implicit.

Figure: Real position y (mm) in blue and y_d (mm) in red, under $h = 2$ms and $h = 15$ms for $\mathbf{G} = 10^5$.

h	2ms	5ms	10ms	15ms
Explicit	1.7838e+03	904.1336	844.2871	1.4462e+03
Saturation	1.6527e+03	914.4627	838.3387	1.6821e+03
Implicit	1.6452e+03	657.6504	428.0244	196.0669

(a) $G = 10^4$

h	2ms	5ms	10ms	15ms
Explicit	2.5724e+03	1.7742e+03	1.6081e+03	2.5070e+03
Saturation	2.5691e+03	2.0749e+03	2.1638e+03	2.5756e+03
Implicit	1.6360e+03	650.2710	480.1660	228.8022

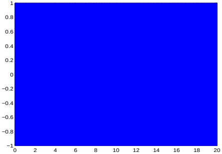
(b) $G = 10^5$

Table: Variation of position error e. The variation is approximated by the quantity $Var_{[a,b]}(f) = \sum_{i=0}^{N-1} |f(t_{i+1}) - f(t_i)|$ where t_i are the sampling times (compare columnwise).

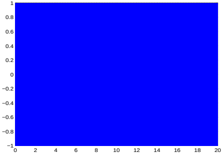
Comments on output y behaviour:

1. For very small sampling time h, and $G = 10^4$, all three methods yield same results.
2. When G is multiplied by 10, the implicit method gives the same output while the explicit and satutation have increasing chattering. This is clearly seen in Table 1 where the implicit method output has much smaller variation for $h \geq 5$ ms.
3. When h is multiplied by 15, the explicit and saturation methods have an output that chatters while the implicit method gives smooth (but slightly delayed) ouput.

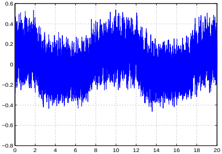
The switching function is the *selection* of the set-valued controller.



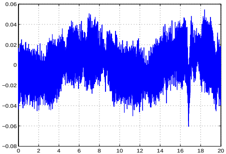
(a) Explicit. $sgn(\sigma_k)$. $G = 10^4$, $h = 2$ms.



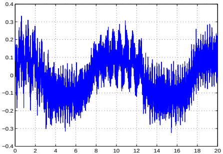
(b) Saturation. $sat(\sigma_k)$. $G = 10^4$, $h = 2$ms.



(c) Implicit. $sgn(\sigma_{k+1})$. $G = 10^4$, $h = 2$ms.

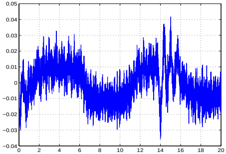


(d) Implicit. $sgn(\sigma_{k+1})$. $G = 10^5$, $h = 2$ms.

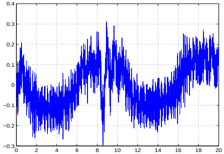


(e) Implicit. $sgn(\sigma_{k+1})$. $G = 10^4$, $h = 5$ms.

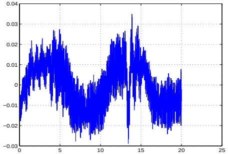
Figure: Explicit ($sgn(s_k)$), saturation ($sat(s_k)$) and implicit ($sgn(s_{k+1})$) algorithms.



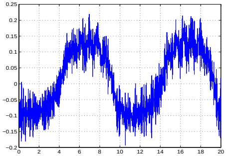
(a) Implicit. $sgn(\sigma_{k+1})$. $G = 10^5$, $h = 5$ms.



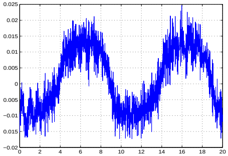
(b) Implicit. $sgn(s_{k+1})$. $G = 10^4$, $h = 10$ms.



(c) Implicit. $sgn(\sigma_{k+1})$. $G = 10^5$, $h = 10$ms.



(d) Implicit. $sgn(\sigma_{k+1})$. $G = 10^4$, $h = 15$ms.



(e) Implicit. $sgn(\sigma_{k+1})$. $G = 10^5$, $h = 15$ms

Figure: Switching function: explicit ($sgn(s_k)$), saturation ($sat(s_k)$) and implicit ($sgn(s_{k+1})$) algorithms.

h	2ms	5ms	10ms	15ms
Explicit	(-1.000, 1.000)	(-1.000, 1.000)	(-1.000, 1.000)	(-1.000, 1.000)
Saturation	(-1.000, 1.000)	(-1.000, 1.000)	(-1.000, 1.000)	(-1.000, 1.000)
Implicit	(-0.4635, 0.5385)	(-0.3247, 0.3338)	(-0.2969, 0.3117)	(-0.1935, 0.2194)

(a) Range of the switching function.

h	2ms	5ms	10ms	15ms
Explicit	6926	2822	2258	1936
Saturation	6.6197e+03	2.7224e+03	2.2199e+03	2008
Implicit	1.8416e+03	357.9547	211.4038	79.1096

(b) Variation of the switching function.

Table: Switching function, gain $G = 10^4$.

h	2ms	5ms	10ms	15ms
Explicit	(-1.000, 1.000)	(-1.000, 1.000)	(-1.000, 1.000)	(-1.000, 1.000)
Saturation	(-1.000, 1.000)	(-1.000, 1.000)	(-1.000, 1.000)	(-1.000, 1.000)
Implicit	(-0.0606, 0.0545)	(-0.0360, 0.0417)	(-0.0289, 0.0349)	(-0.0173, 0.0247)

(a) Range of the switching function.

h	2ms	5ms	10ms	15ms
Explicit	2980	2050	1932	1836
Saturation	2.3486e+03	1.9858e+03	1902	1860
Implicit	183.1965	34.7510	25.2005	8.1039

(b) Variation of the switching function.

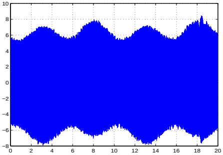
Table: Switching function, gain $G = 10^5$.

Comments on the switching functions behaviour:

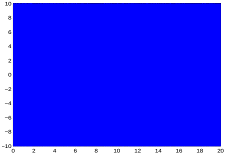
1. The explicit and saturation methods yield high-frequency and maximal magnitude bang-bang signals.
2. The implicit method yields smaller magnitude signal, whose shape is almost constant for $h > 5$ ms, for both gains G (recall the magnitude of the switching function is divided by 10 when G is multiplied by 10): this confirms theoretical findings about controller-gain insensitivity.
3. The implicit controller continues to behave very well for large sampling times $h = 15$ ms:

   $\rightsquigarrow$ *This is thought to be a very nice property for applications together with gain insensitivity.*

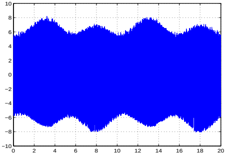
   $\rightsquigarrow$ *This means that the designer can augment the gain G (if for instance the disturbance increases) without creating chattering and input increase as is the case with the explicit and saturation methods.*



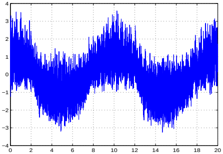
(a) Explicit. $G = 10^4$, $h = 2$ms.



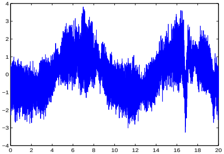
(b) Explicit and Saturation. $G = 10^5$, $h = 2$ms.



(c) Saturation. $G = 10^4$, $h = 2$ms

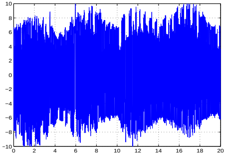


(d) Implicit. $G = 10^4$, $h = 2$ms.

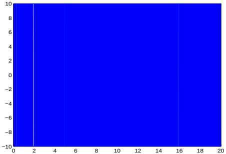


(e) Implicit. $G = 10^5$, $h = 2$ms.

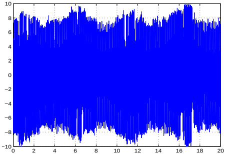
Figure: Total control u: explicit, saturation and implicit methods.



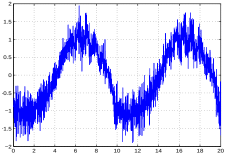
(a) Explicit. $G = 10^4$, $h = 15$ms.



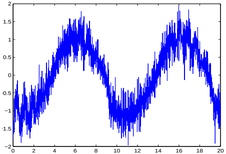
(b) Explicit and Saturation. $G = 10^5$, $h = 15$ms.



(c) Saturation. $G = 10^4$, $h = 15$ms



(d) Implicit. $G = 10^4$, $h = 15$ms.



(e) Implicit. $G = 10^5$, $h = 15$ms.

Figure: Total control u: explicit, saturation and implicit methods.

h	2ms	5ms	10ms	15ms
Explicit	(-7.8876 8.4594)	(-8.1550 8.6118)	(-8.7349 8.3970)	(-10 10)
Saturation	(-8.0737 8.1963)	(-7.9095 8.0899)	(-8.5541 8.7543)	(-10 10)
Implicit	(-3.2500 3.5871)	(-1.9990 2.6204)	(-1.9399 2.1267)	(-1.8990 1.9484)

(a) range of u.

h	2ms	5ms	10ms	15ms
Explicit	4.1102e+04	1.7731e+04	1.3816e+04	1.2759e+04
Saturation	4.0209e+04	1.6864e+04	1.3838e+04	1.3671e+04
Implicit	9.5190e+03	1.5731e+03	963.2736	609.5058

(b) variation of u.

Table: Comparisons of u when $G = 10^4$.

h	2ms	5ms	10ms	15ms
Explicit	(-10 10)	(-10 10)	(-10 10)	(-10 10)
Saturation	(-10 10)	(-10 10)	(-10 10)	(-10 10)
Implicit	(-3.2541 3.8092)	(-2.0772 2.6066)	(-2.0325 2.3656)	(-1.9642 1.9461)

(a) range of u.

h	2ms	5ms	10ms	15ms
Explicit	29800	20500	19320	18360
Saturation	2.3516e+04	1.9846e+04	19020	18600
Implicit	9.3245e+03	1.5389e+03	1.1560e+03	629.0904

(b) variation of u.

Table: Comparisons of u when $G = 10^5$.

Comments on the total controllers u behaviour:

1. The implicit method provides much better input than the explicit and saturation ones, both in terms of:
   1.1 input magnitude divided by 5 for $h \geq 5$ ms.
   1.2 input variation (chattering) divided by 15 to 30 for $h \geq 5$ ms.
2. Insensitivity of u with respect to G.
3. The behaviour improves when G increases (which may be counterintuitive).

Sliding-variable differentiators:

A last comment: the sliding variable in (24) is obtained by direct twice differentiation of y with first-order filters to get $\dot{y}$ and $\ddot{y}$ ("dirty" differentiation).

- The tuning of these filters can influence significantly the closed-loop behaviour.
- Bandwidth limitations of these filters can explain the deterioration of the performance of the implicit method for very small $h = 2$ ms

Let us now study the inverted pendulum on a cart (setup from CRIsTAL, Ecole Centrale de Lille, France):

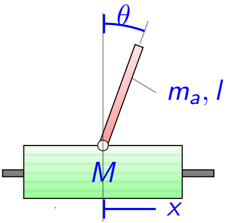


Figure: Inverted pendulum on a cart.

We use the following linearized model around the unstable equilibrium $\mathbf{x}_{\mathrm{eq}} = (0\ 0\ 0\ 0)^T$, with $\mathbf{x} = (x\ \dot{x}\ \theta\ \dot{\theta})^T$:

$$\dot{\mathbf{x}} = A\mathbf{x} + Bu, \quad y = H\mathbf{x}$$

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & -\frac{m_a}{M}g & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & \frac{M+m_a}{Ml}g & 0 \end{pmatrix}, B = \begin{pmatrix} 0 \\ \frac{a}{M} \\ 0 \\ -\frac{a}{Ml} \end{pmatrix}, H^T = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

with $M = 3.9249\mathrm{kg}$, $m_a = 0.2047\mathrm{kg}$ the mass of the cart and the pendulum, $l = 0.2302\mathrm{m}$ is the length of the pendulum, $a = 25.3\mathrm{N/V}$ is the motor gain.

The control input u is proportional to the input voltage of the linear motor.

1. The control objective is to maintain the pendulum at the unstable equilibrium $\mathbf{x}_{\mathrm{eq}}$.
2. The sliding surface was designed using an LMI procedure, such that on the sliding manifold the non-zero eigenvalues of the closed-loop system are in a cone in the left-hand complex plane. This criteria is expected to reduce the oscillations on the sliding surface.
3. The experiments were done with an initial position close to the unstable equilibrium in order to avoid the additional complexity of a switching logic between a local sliding mode controller and global controller.
4. Therefore the reaching phase is short or nonexistent and the closed-loop system is mostly in the discrete-time sliding phase with the controller.
5. With the sampling period set to $20\mathrm{ms}$, the scalar CB^* is equal to 0.1978.

Implicit (top) and explicit (bottom) controllers with $h = 20\text{ms}$, $\alpha = 1$.

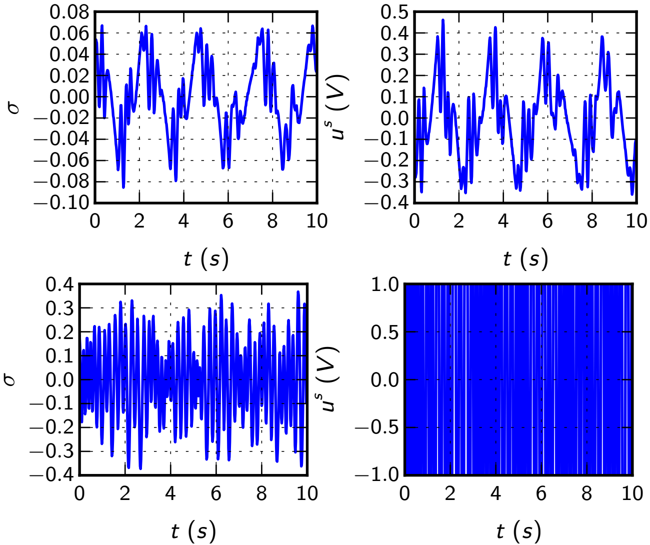
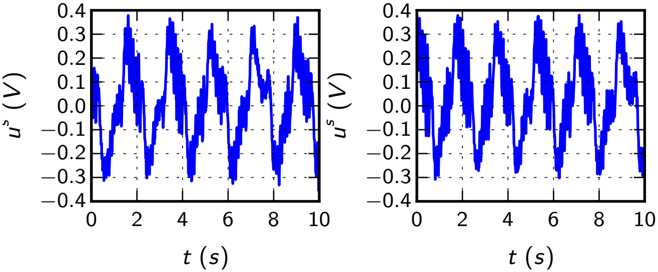


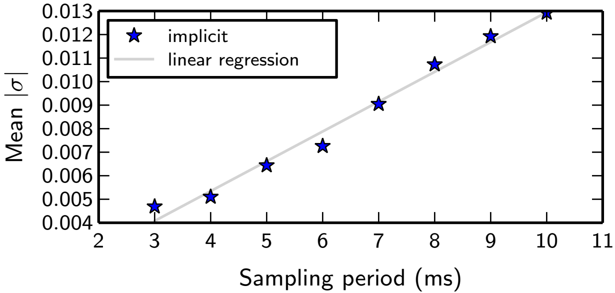
Table: Control input and sliding variable variations with both the implicit and explicit controller.

Controller	$\mathrm{Var}_0^{10}(\bar{u})$	$\mathrm{Var}_0^{10}(\sigma)$
Implicit	96.24	3.10
Explicit	1332.89	44.74

Control input values with 2 different gains: $\alpha = 1$ on the left and $\alpha = 3$ on the right; $h = 7\mathrm{ms}$.



Precision with respect to the sampling period:



$\rightsquigarrow$ *linearity with respect to h.*

Some analytical results about the precision of the implicit algorithm, when the ZOH discretizatin is used:

Lemma (Precision of the implicit method)

Let the closed-loop system be in the discrete-time sliding phase. In the unperturbed case, if the discontinuous part u^s of the control is discretized using an implicit scheme, then the total discretization error $\varepsilon_k = ||\sigma_{k+1}||$ has the same order as the discretization error $\Delta\bar{\sigma}_k$ on u^{eq} (that is h^2 if $\bar{u}^{eq}$ is discretized explicitly or implicitly, and h^3 if it is discretized with a midpoint method). If there is a matched perturbation, then the order is 1 *and this increase of the order is due to the perturbation.*

$\rightsquigarrow$ *This is not at all the case for the explicit method, [Huber et al, TACON 2016, Lemma 10].*

- Both sets of experiments show that the implicit method allows to:
  1. Drastically decrease the controller chattering and magnitude,
  2. Significantly decrease the output chattering and magnitude (much better precision),

  without changing the set-valued control structure (one gain).
- recover the nice and powerful properties of the continuous-time SMC: insensitivity w.r.t. the control gain in the sliding phases, finite-time convergence.
- design large sampling periods without deteriorating the closed-loop behaviour.

$\rightsquigarrow$ *All done for first-order "classical" SMC. Need for further comparisons with twisting (to come later), super-twisting, disturbance estimation techniques,* etc.

ROBUST CONTROL OF LAGRANGIAN SYSTEMS WITH UNCERTAINTIES

Let us deal with Lagrangian systems:

$$\boxed{M(q)\ddot{q} + C(q,\dot{q})\dot{q} + G(q) + F(t,q,\dot{q}) = \tau}$$

where:

- $q, \dot{q}, \ddot{q} \in \mathbb{R}^n$ are the vectors of generalized positions, velocities and accelerations, respectively.
- $M(q) \in \mathbb{R}^{n\times n}$ denotes the inertia matrix of the system, $M(q) = M(q)^\top > 0$,
- $C(q,\dot{q})\dot{q} \in \mathbb{R}^n$ represents the centripetal and Coriolis forces,
- $G(q) \in \mathbb{R}^n$ is the vector related with gravitational forces,
- $F(t,q,\dot{q}) \in \mathbb{R}^n$ accounts for unmodeled dynamics and external disturbances.
- $\tau \in \mathbb{R}^n$ represents the control input forces.

Let us state some standard properties and assumptions:

Property

The matrices $M(q)$ and $C(q, \dot{q})$ satisfy for all differentiable functions q:

$$\frac{d}{dt} M(q(t)) = C(q(t), \dot{q}(t)) + C^{\top}(q(t), \dot{q}(t)).$$

Notice that this previous property implies that $\dot{M}(q) - 2C(q, \dot{q})$ is skew-symmetric.

Assumption

1. *The matrices* $M(q)$*,* $C(q,\dot{q})$ *together with the vectors* $G(q)$ *and* $F(t,q,\dot{q})$ *satisfy the following inequalities for all* $(t,q,\dot{q}) \in \mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{R}^n$ *and some positive bounded constants* k_1, k_2, k_C, k_G *and* k_F*:*

$$0 < k_1 \leq \|M(q)\|_m \leq k_2, \;\; \|C(q,\dot{q})\|_m \leq k_C\|\dot{q}\|,$$
$$\|G(q)\| \leq k_G\|q\|, \;\; \|F(t,q,\dot{q})\| \leq k_F.$$

2. *There exists a constant* k_3 *such that for all* $x, y \in \mathbb{R}^n$

$$\|M(x) - M(y)\|_m \leq k_3\|x - y\|.$$

3. *The function* $h : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ *defined by* $h(x_1,x_2,x_3) := C(x_1,x_2)x_3$ *is locally Lipschitz.*

(Continued)

Assumption

1. *The function* $F(t, x_1, x_2)$ *is continuous in* t*, uniformly locally Lipschitz in* (x_1, x_2)*.*
2. *The function* $G(\cdot)$ *is Lipschitz continuous and satisfies* $0 = G(0) \leq G(x)$ *for all* $x \in \mathbb{R}^n$*.*

$\rightsquigarrow$ *These will be used both for well-posedness of the set-valued dynamics, and for the control design.*

The control law takes the form:

$$\boxed{\tau(q,\dot{q}) = \hat{M}(q)\ddot{q}_{\mathrm{r}} + \hat{C}(q,\dot{q})\dot{q}_{\mathrm{r}} + \hat{G}(q) - K_p\tilde{q} + u}$$

where $\dot{q}_{\mathrm{r}} = \dot{q}_{\mathrm{d}} - \Lambda\tilde{q}$, $K_p \in \mathbb{R}^{n\times n}$, $K_p = K_p^\top > 0$, and

$$\boxed{-u \in \gamma(\sigma,\tilde{q})\partial\Phi(\sigma)}$$

where the function $\gamma : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}_+$ is locally Lipschitz continuous and is specified later. Additionally, Φ is selected as:

Assumption

The function $\Phi \in \Gamma_0(\mathbb{R}^n)$ *satisfies* $0 = \Phi(0) \leq \Phi(w)$ *for all* $w \in \mathbb{R}^n$. *Moreover, we have that* $0 \in \mathrm{int}\partial\Phi(0)$.

Examples: $\Phi = ||x||_1 = \sum_{i=1}^n |x_i|$. Or $\Phi = ||x||_\infty = \max_i |x_i|$. or $\Phi = ||x||_2$.

The nominal terms are designed to satisfy the following:

Assumption

The matrices $\hat{M}(q)$, $\hat{C}(q,\dot{q})$ together with the vector $\hat{G}(q)$ satisfy the following inequalities for all $(t,q,\dot{q}) \in \mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{R}^n$ and some known positive constants $\hat{k}_1, \hat{k}_2, \hat{k}_C$ and $\hat{k}_G$

$$0 < \hat{k}_1 \leq \|\hat{M}(q)\|_m \leq \hat{k}_2, \qquad \|\hat{C}(q,\dot{q})\|_m \leq \hat{k}_C\|\dot{q}\|,$$
$$\|\hat{G}(q)\| \leq \hat{k}_G\|q\|.$$

The closed-loop system is rewritten as:

$$\boxed{\begin{array}{l} M(q)\dot{\sigma} + C(q,\dot{q})\sigma + K_p\tilde{q} + \xi(t,\sigma,\tilde{q}) = u \\ \dot{\tilde{q}} = \sigma - \Lambda\tilde{q} \\ \\ -u \in \gamma(\sigma,\tilde{q})\partial\Phi(\sigma) \end{array}}$$

where thanks to the above assumptions:

Proposition

The function $\xi(t,\sigma,\tilde{q})$ satisfies

$$\|\xi(t,\sigma,\tilde{q})\| \leq \beta(\sigma,\tilde{q}),$$

where $\beta(\sigma,\tilde{q}) = c_1 + c_2\|\sigma\| + c_3\|\tilde{q}\| + c_4\|\tilde{q}\|\|\sigma\| + c_5\|\tilde{q}\|^2$, for bounded positive constants c_i, $i = 1,\ldots,5$.

Theorem

Let the above assumptions hold and in addition there exists $\alpha > 0$ such that, $\Phi(\cdot) \geq \alpha\|\cdot\|$. Then, there exists a solution $\sigma : [0, +\infty) \to \mathbb{R}^n$, $\tilde{q} : [0, +\infty) \to \mathbb{R}^n$ of (25) for every $(\sigma_0, \tilde{q}_0) \in \mathbb{R}^n \times \mathbb{R}^n$, whenever $\frac{\alpha}{2}\gamma(\sigma, \tilde{q}) \geq \beta(\sigma, \tilde{q})$. The notion of solution is taken in the following sense:

- *$\tilde{q}$ is continuous with derivative $\dot{\tilde{q}}$ continuous and bounded in bounded sets.*
- *σ is continuous and bounded in bounded sets, $\dot{\sigma} \in L_\infty([0, +\infty); \mathbb{R}^n)$.*
- *Equations (25) are satisfied for almost all $t \in [0, +\infty)$, $\sigma(0) = \sigma_0$ and $\tilde{q}(0) = \tilde{q}_0$.*

$\rightsquigarrow$ *The result continues to hold with a constant γ.*

Theorem (Finite-time convergence)

Consider the system (25). *Let the assumptions of Theorem 9 hold. Set* $\gamma(\sigma, \tilde{q}) = (2\beta(\sigma, \tilde{q}) + \delta)/\alpha$, *where* $\delta > 0$ *is constant and* β *is defined as above. Then, the sliding surface* $\sigma = 0$ *is reached in finite time.*

Theorem (Practical stability)

Let the assumptions of Theorem 9 hold. Consider system (25) *with the multivalued control law* $u \in -\gamma\partial\Phi(\sigma)$ *and consider a compact set* $W_{R,K_p} = \{(\sigma, \tilde{q}) \in \mathbb{R}^n \times \mathbb{R}^n \mid \frac{1}{2}\sigma^\top M(q)\sigma + \frac{1}{2}\tilde{q}^\top K_p\tilde{q} \leq R\}$ *with* $R > 0$ *fixed. The origin of the closed-loop system is semi-globally asymptotically stable. Moreover, the basin of attraction contains* W_{R,K_p} *whenever*

$$\gamma > \frac{R_\xi}{\alpha}, \tag{25}$$

with $R_\xi = \max_{(\sigma,\tilde{q})\in W_{R,K_p}} \beta(\sigma, \tilde{q})$.

Let us start with the following Euler discretization of the plant:

$$\boxed{\begin{array}{l} M(q_k)\frac{\dot{q}_{k+1}-\dot{q}_k}{h} + C(q_k,\dot{q}_k)\dot{q}_{k+1} + G(q_k) + F(t_k,q_k,\dot{q}_k) = \tau_k \\ \\ q_{k+1} = q_k + h\dot{q}_k \end{array}}$$

Since we cannot apply the ZOH discretization we choose this Euler method, design a controller from it and then prove some convergence towards the continuous time system.

Property (Skew-symmetry of the nominal terms)

The matrices $\hat{M}(q)$ and $\hat{C}(q, \dot{q})$ satisfy

$$\frac{d}{dt}\hat{M}(q(t)) = \hat{C}(q(t), \dot{q}(t)) + \hat{C}^\top(q(t), \dot{q}(t))$$

Lemma (Approximate skew-symmetry property)

For any $k \geq 0$ we have

$$\begin{aligned} \hat{M}_{k+1} - \hat{M}_k &= h\hat{C}_k + h\hat{C}_k^\top + \hat{\epsilon}_k \\ M_{k+1} - M_k &= hC_k + hC_k^\top + \epsilon_k \end{aligned}$$

where $\epsilon_k, \hat{\epsilon}_k \in \mathbb{R}^{n\times n}$ are $o(h)$ ('little-o') matrix functions, i.e.,

$$\lim_{h\downarrow 0} \frac{\|\hat{\epsilon}_k\|_m}{h} = \lim_{h\downarrow 0} \frac{\|\epsilon_k\|_m}{h} = 0.$$

We propose the control law τ_k as

$$\boxed{\begin{aligned}&\tau_k = \hat{M}_k \frac{\dot{q}^r_{k+1} - \dot{q}^r_k}{h} + \hat{C}_k \dot{q}^r_{k+1} + \hat{G}_k + u_{k+1}\\[2ex] &-u_{k+1} \in K_\sigma \hat{\sigma}_{k+1} + \gamma \partial \Phi(\hat{\sigma}_{k+1})\\[2ex] &q^r_{k+1} = q^r_k + h\dot{q}^r_k\end{aligned}}$$

where $\dot{q}^r_k = \dot{q}^{\mathrm{d}}_k - \Lambda \tilde{q}_k$. After some simple algebraic manipulations, the closed-loop system is obtained as

$$\boxed{\begin{aligned}&M_k\sigma_{k+1} - M_k\sigma_k + hC_k\sigma_{k+1} = -h\xi_k + hu_{k+1},\\[2ex] &-u_{k+1} \in K_\sigma \hat{\sigma}_{k+1} + \gamma \partial \Phi(\hat{\sigma}_{k+1})\\[2ex] &\tilde{q}_{k+1} = (I - h\Lambda)\,\tilde{q}_k + h\sigma_k\end{aligned}}$$

The term ξ_k satisfies a similar property as its continuous-time counterpart: $\|\xi(t_k, \sigma_k, \tilde{q}_k)\| \leq \beta(\sigma_k, \tilde{q}_k)$.

Since the “disturbance” $\xi(t_k, \sigma_k, \tilde{q}_k)$ is unknown we proceed as in the foregoing cases to calculate the controller: with a nominal unperturbed auxiliary system which serves as a generalized equation (GE).

$\rightsquigarrow$ *However this time this GE is less easy because of the nonlinearities.*

Plant+pre-feedback:

$$\boxed{\begin{array}{l} M_k\sigma_{k+1} - M_k\sigma_k + hC_k\sigma_{k+1} + hK_\sigma\hat{\sigma}_{k+1} - h\xi_k = -h\gamma\zeta_{k+1} \\ \tilde{q}_{k+1} = (I - h\Lambda)\,\tilde{q}_k + h\sigma_k \end{array}}$$

Nominal unperturbed system for control computation:

$$\boxed{\begin{array}{l} \hat{M}_k\hat{\sigma}_{k+1} - \hat{M}_k\sigma_k + h\hat{C}_k\hat{\sigma}_{k+1} + hK_\sigma\hat{\sigma}_{k+1} = -h\gamma\zeta_{k+1} \\ \zeta_{k+1} \in \partial\Phi(\hat{\sigma}_{k+1}) \end{array}}$$

When all disturbances and uncertainties vanish, then $\hat{\sigma}_k = \sigma_k$ provided $\hat{\sigma}_0 = \sigma_0$.

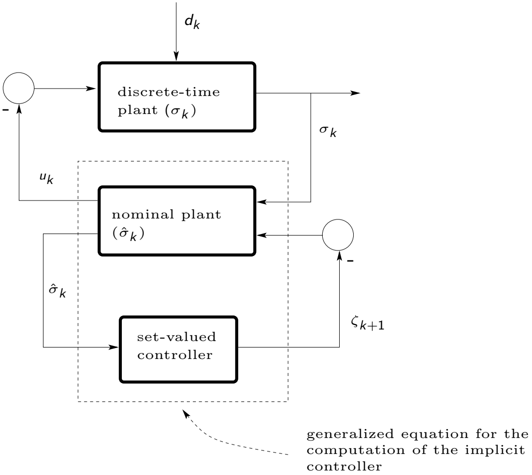


Figure: The generic block diagram for the implicit method.

First step: well-posedness of the general scheme, i.e., we can compute a selection of the multivalued controller in a unique fashion, using only the information available at the time step k.

We can rewrite equivalently the GE as:

$$(\hat{M}_k + h\hat{C}_k + hK_\sigma)\hat{\sigma}_{k+1} - \hat{M}_k\sigma_k \in -h\gamma\partial\Phi(\hat{\sigma}_{k+1})$$

Equivalently, as a *variational inequality (VI) of the second kind*:

$$\boxed{\begin{array}{l} \left\langle \hat{\mathcal{A}}_k\hat{\sigma}_{k+1} - \hat{M}_k\sigma_k, \eta - \hat{\sigma}_{k+1}\right\rangle + h\gamma\Phi(\eta) - h\gamma\Phi(\hat{\sigma}_{k+1}) \geq 0 \\ \text{for all } \eta \in \mathbb{R}^n, \\ \text{with } \hat{\mathcal{A}}_k \triangleq \left(\hat{M}_k + h\hat{C}_k + hK_\sigma\right) \end{array}}$$

Why using the VI formalism ? Because we can use this result:

Lemma

Let $f \in \Gamma_0(X)$ and let $\mathcal{A} : X \to X$ be a continuous and strongly monotone operator. That is, for any $x_1, x_2 \in X$,

$$\langle \mathcal{A}(x_1) - \mathcal{A}(x_2), x_1 - x_2 \rangle \geq \alpha \|x_1 - x_2\|$$

for some $\alpha > 0$. Then, for each $v \in X$, there exists a unique solution $x \in X$ to the variational inequality

$$\langle \mathcal{A}x - v, \eta - x \rangle + f(\eta) - f(x) \geq 0 \quad \text{for all } \eta \in X.$$

See for instance [Facchinei and Pang, 2003].

Then skipping some intermediate steps, it follows that the selection of the control value can be obtained as:

$$\boxed{\begin{aligned} \zeta_{k+1} &= -\frac{1}{h\gamma}(\hat{\mathcal{A}}_k\hat{\sigma}_{k+1} - \hat{M}_k\sigma_k) \\ \hat{\sigma}_{k+1} &= \mathrm{Prox}_{\mu h\gamma\Phi}((I - \mu\hat{\mathcal{A}}_k)\hat{\sigma}_{k+1} + \mu\hat{M}_k\sigma_k) \end{aligned}}$$

where $\mu > 0$ is such that $0 < \hat{\mathcal{A}}_k + \hat{\mathcal{A}}_k^\top - \mu\hat{\mathcal{A}}_k^\top\hat{\mathcal{A}}_k$ (to guarantee contraction property).

The operator $\mathrm{Prox}_{\mu h\gamma\Phi}(\cdot)$ is called the *proximity operator* of $\mu h\gamma\,\Phi(\cdot)$.

Examples of $\mathrm{Prox}_{\lambda f}$**:**

- l_1-norm: $f(x) = ||x||_1$: $\mathrm{Prox}_{\lambda f}(x) = x - \mathrm{proj}([-\lambda, \lambda]^m; x)$
- l_2-norm: $f(x) = ||x||_2$: $\mathrm{Prox}_f(x) = x - \mathrm{proj}(B(0,1); x) = \begin{cases} \left(1 - \frac{1}{||x||_2}\right) x & \text{if } ||x|| > 1 \\ 0 & \text{if } ||x|| \leq 1 \end{cases}.$

It has the property that

$$p = \mathrm{Prox}_f(x) \Leftrightarrow x - p \in \partial f(p) \Leftrightarrow p \in (I + \partial f)^{-1}(x).$$

One important point is how to solve efficiently the problem

$$\hat{\sigma}_{k+1} = \mathrm{Prox}_{\mu h \gamma \Phi}((I - \mu \hat{\mathcal{A}}_k)\hat{\sigma}_{k+1} + \mu \hat{M}_k \sigma_k) :$$

- the semi-smooth Newton method [Facchinei-Pang 2001]. For control applications this method may be too time-consuming since it involves the computation of inverse matrices and proximal maps of composite functions.
- In contrast, the simple method of successive approximations can quickly find the fixed point.

Example: set $\Phi(x) = \alpha\|x\|_1$, $\partial\Phi(x) = (\mathrm{sgn}(x_1), \ldots, \mathrm{sgn}(x_n))^T$.
The successive approximations method is as follows:

1. Set $\mu > 0$ small enough such that $0 < \hat{\mathcal{A}}_k + \hat{\mathcal{A}}_k^\top - \mu\hat{\mathcal{A}}_k^\top\hat{\mathcal{A}}_k$ holds.
2. Set $j = 0$ and set $x^0 \in \mathbb{R}^n$.
3. Compute x^{j+1} as

$$v^j = (I - \mu\hat{\mathcal{A}}_k)x^j + \mu\hat{M}_k\sigma_k,$$

$$x^{j+1} = v^j - \mu\,\mathrm{Proj}_{[-c,c]^n}\left(\frac{v^j}{\mu}\right),$$

   where $c = h\gamma\alpha$ and the set $[-c, c]^n$ represents the n-cube in $\mathbb{R}^n$ centered at the origin with edge length equal to $2c$.

4. If $\|x^{j+1} - x^j\| > \varepsilon$, then increase j and go to step 3. Else, set $\hat{\sigma}_{k+1} = x^{j+1}$ and stop.

The scalar ε represents the precision of the algorithm.

Second step: Stability of the discrete-time closed loop system.

Theorem (Parametric uncertainty)

Let all the above assumptions hold. Then, there exist constants $\hat{r}_\sigma > 0$ and $h^* > 0$, $\delta^* > 0$, such that for all $h \in (0, \min\{\delta^*, h^*\}]$, the origin of the discrete-time closed-loop system is semi-globally practically stable whenever γ and α satisfy

$$\gamma\alpha > \max\left\{\frac{2\hat{k}_2}{\hat{k}_1}\bar{\beta}\left(1+\frac{\bar{\beta}}{\hat{k}_1\hat{r}_\sigma}\right), 2\hat{k}_2\sqrt{\frac{\hat{k}_2}{\hat{k}_1}}\left(\hat{r}_\sigma+\frac{2\mathcal{F}}{\hat{k}_1}\right)\right\}. \quad (27)$$

for some constants $\bar{\beta}$ and $\mathcal{F}$. Moreover, $\hat{\sigma}_k$ reaches the origin in a finite number of steps k^*, and $\hat{\sigma}_k = 0$ for all $k \geq k^* + 1$.

Comments:

1. The constant δ^* guarantees that the skew-symmetry property is well approximated (the higher order terms are small enough).
2. The controller is less easy to calculate, the conditions for stability are more stringent, and the stability is less good: but we deal with nonlinear systems with parametric uncertainties and external disturbances...
3. The stability proof is led with the Lyapunov-like functions $V_{1,k} = \hat{\sigma}_k^\top \hat{M}_k \hat{\sigma}_k$ and $V_{2,k} = \sigma_k^\top \hat{M}_k \sigma_k$.
4. The framework allows for co-dimension ≥ 2 attractive surfaces (and with uniqueness of solutions).

Third step: Convergence of the discrete-time solutions towards the continous-time ones.

Let the functions

$$\sigma_h(t) = \sigma_{k+1} + \frac{t_{k+1} - t}{h}(\sigma_k - \sigma_{k+1})$$
$$\tilde{q}_h(t) = \tilde{q}_{k+1} + \frac{t_{k+1} - t}{h}(\tilde{q}_k - \tilde{q}_{k+1})$$

for all $t \in [t_k, t_{k+1})$, be the piecewise-linear approximations of σ_k and $\tilde{q}_k$ respectively.

Theorem (Convergence of the discrete-time solutions)

Let $(\sigma_k, \tilde{q}_k)$ be a solution of the closed-loop discrete-time system. Then, $(\sigma_h, \tilde{q}_h)$ converges to $(\sigma, \tilde{q})$ as the sampling time h decreases to zero, where $(\sigma, \tilde{q})$ is a solution of

$$\begin{cases} M(q(t))\dot{\sigma}(t) + C\left(q(t), \dot{q}(t)\right)\sigma(t) + K_\sigma \sigma(t) + \xi(t, \sigma(t), \tilde{q}(t)) = -\gamma\zeta(t) \\ \zeta(t) \in \partial\Phi\left(\sigma(t)\right) \\ \dot{\tilde{q}}(t) = \sigma(t) - \Lambda\tilde{q}(t) \end{cases}$$

with $\sigma(0) = \sigma_0$ and $\tilde{q}(0) = \tilde{q}_0$.

$\rightsquigarrow$ *This results guarantees that despite the plant Euler discretization is a non-exact discretization, the overall discrete-time design makes sense.*

Fourth step: Numerical simulations.

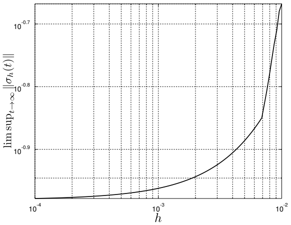


Figure: Evolution of the $\limsup_{t\to\infty}\|\sigma_h(t)\|$ of the sliding variable σ_h as a function of h in a logarithmic scale.

Depicts how the norm of the sliding variable σ_h, associated with the continuous plant/discrete controller setting, evolves as a function of the sampling time $h > 0$.

$\rightsquigarrow$ *The order of convergence is not constant and, moreover, it tends to zero as h decreases to zero.*

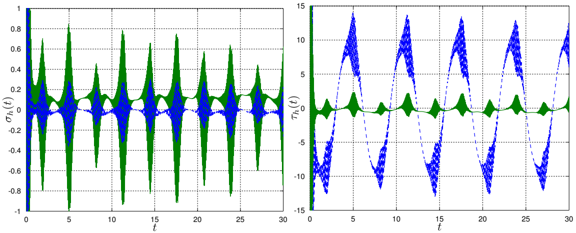


Figure: Implicit method: sliding variable σ_h (left) and control input τ_h (right) with $h = 10^{-2}$ s.

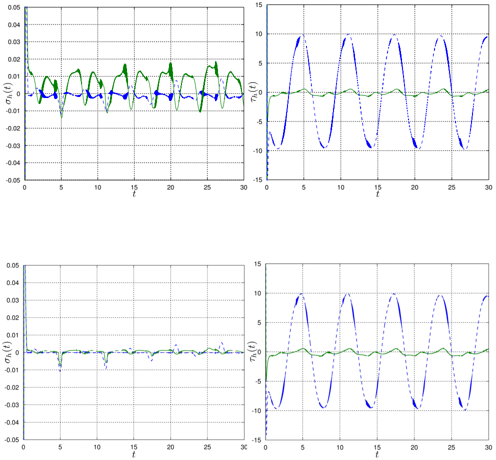


Figure: Implict method: sliding variable σ_h (left) and control input τ_h (right) with $h = 10^{-3}$ s (top) $h = 10^{-4}$ s (bottom).

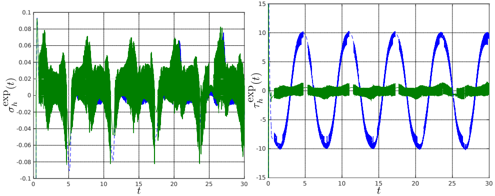


Figure: Explicit method: sliding variable (left) and control input (right) with $h = 10^{-3}$ s.

$\rightsquigarrow$ *Instability with $h = 10^{-2}$s and the explicit method.*

So we recover here that the implicit method allows larger sampling times.

Fifth step: Experimental validations.

Not performed yet !

IMPLICIT DISCRETIZATION OF THE TWISTING ALGORITHM: EXPERIMENTAL RESULTS

Fact: the twisting algorithm discretized with an explicit Euler method, usually yields chattering (theoretical analysis [Yan 2016], experimental results [Huber 2016]).

Objective: study the implicit implementation of the twisting algorithm on the above electropneumatic setup.

Figure: Photography of the electropneumatic system (LS2N, Ecole Centrale de Nantes, France).

Let us consider:

$$\ddot{\sigma}(x,t) = a(x,t) + b(x,t)u \tag{28}$$

with the following bounds: for all $(x,t) \in \mathbb{R}^n \times \mathbb{R}_+$,

$$0 \leq K_m \leq |b(x,t)| \leq K_M \quad \text{and} \quad |a(x,t)| \leq K_a$$

The control law for the twisting controller is

$$u \in -r_1 \operatorname{Sgn}(\sigma) - r_2 \operatorname{Sgn}(\dot{\sigma}) \tag{29}$$

and with the conditions

$$\begin{cases} (r_1 + r_2)K_m - K_a > (r_1 - r_2)K_M + K_a \\ (r_1 - r_2)K_m > K_a \end{cases}$$

the state of the closed-loop system (28) and (29) converges to the origin in finite time.

We follow the convention of using $G = r_1$ and $\beta = r_2/r_1$, instead of r_1 and r_2.

The implicit discretization is:

$$u_{k+1} \in -G\,\mathrm{Sgn}(\sigma_{k+1}) - \beta G\,\mathrm{Sgn}(\dot{\sigma}_{k+1})$$

whereas the explicit discretization yields:

$$u_k = -G\,\mathrm{sgn}(\sigma_k) - \beta G\,\mathrm{sgn}(\dot{\sigma}_k) \tag{30}$$

Note that the relation in (30) *is not an inclusion since the right-hand side is a given singleton at time* t_k.

The discrete-time dynamics of Σ with state $(\sigma, \dot{\sigma})$ is supposed to be:

$$\boxed{\widetilde{\Sigma}_{k+1} = A_k^d \Sigma_k + F_k^d + B_k^d \lambda} \tag{31}$$

where $\lambda = (\lambda_1\ \lambda_2)^T$, $\lambda_1 \in -\mathrm{Sgn}(\sigma_{k+1})$, $\lambda_2 \in -\mathrm{Sgn}(\dot{\sigma}_{k+1})$.

$\widetilde{\Sigma}_{k+1}$ *is in general not equal to* $\Sigma(t_{k+1})$ *(except if we use a ZOH method with an LTI plant model).*

We let $u_{k+1} = G(1 \ \ \beta)\lambda$ and thus the value of λ is obtained as the solution of the following generalized equation:

$$\begin{cases} \widetilde{\Sigma}_{k+1} = A_k^d \Sigma_k + F_k^d + B_k^d \lambda \\ \\ \lambda \in -\operatorname{Sgn}(\widetilde{\Sigma}_{k+1}) \end{cases}$$

with unknowns λ and $\widetilde{\Sigma}_{k+1}$. Using the Convex Analysis tools as above we get the generalized equation (GE)

$$0 \in A_k^d \Sigma_k + F_k^d + B_k^d \lambda + \mathcal{N}_{[-1,1]^2}(\lambda)$$

which features only one unknown: λ.

Once again we may use the AVI theory. The GE is an equivalent form of an Affine Variational Inequality (AVI). Solving this AVI consists in:

Find $\lambda \in [-1,1]^2$ such that:

$$\text{for all } w \in [-1,1]^2 \quad (w-\lambda)^T L_k(\lambda) \geq 0 \tag{32}$$

with $L_k \colon \lambda \mapsto A_k^d \Sigma_k + F_k^d + B_k^d \lambda$ an affine map.

Lemma (Implicit controller existence)

The AVI (32) *has always a solution.*

Uniqueness holds but is more tricky, see later.

The dynamics of the electropneumatic experimental setup is

$$\begin{cases} \dot{p}_P = \dfrac{\kappa r T}{V_P(y)}[\varphi_P + \psi_P u - \dfrac{S}{rT} p_P v] \\ \dot{p}_N = \dfrac{\kappa r T}{V_N(y)}[\varphi_N - \psi_N u + \dfrac{S}{rT} p_N v] \\ \dot{v} = \dfrac{1}{M}[S\,(p_P - p_N) - b_v v - F] \\ \dot{y} = v \end{cases} \tag{33}$$

$\rightsquigarrow$ *The position y, the pressures p_P, p_N are available but both the speed v and acceleration are computed using a "dirty" differentiator given by $D(s) = \frac{s}{1+\tau s}$.*

The sliding variable is defined as

$$\sigma = \alpha e + \dot{e} \tag{34}$$

with $\alpha > 0$ a parameter that should becarefully selected. After manipulations we obtain

$$\ddot{\sigma} = \Phi + \Delta\Phi + (\Psi + \Delta\Psi)u$$

and the generalized equation for the implicit controller calculation:

$$\boxed{\begin{array}{l} 0 \in \sigma_k + h\dot{\sigma}_k + \frac{h^2}{2}\left(\Phi_k + G\Psi_k[\lambda_1 + \beta\lambda_2]\right) + \mathcal{N}_{[-1,1]}(\lambda_1) \\ \\ 0 \in \dot{\sigma}_k + h\Phi_k + hG\Psi_k[\lambda_1 + \beta\lambda_2] + \mathcal{N}_{[-1,1]}(\lambda_2) \end{array}}$$

with unknowns λ_1 and λ_2. This is the problem solved to compute the control input value at each time instant t_k.

Proposition (Implicit controller uniqueness)

The implicit twisting controller, defined by the above generalized equation, has a unique solution $\widetilde{\Sigma}_{k+1}$ *and control input value* u_k*. Moreover if* $\widetilde{\Sigma}_{k+1} \neq 0$*, then the pair* (λ_1, λ_2) *is also unique.*

$\rightsquigarrow$ *The same result holds in the continuous-time twisting algorithm, where the selections* $\lambda_1 \in -\operatorname{Sgn}(\sigma)$ *and* $\lambda_2 \in -\operatorname{Sgn}(\dot{\sigma})$ *are uniquely defined, except when* $u = 0$*.*

1. Since λ takes values in a compact convex set, a solution to the AVI with any matrix B_k^d can be computed using the algorithm proposed in [Cao and Ferris 1996], implemented in the INRIA SICONOS software package

   http://siconos.gforge.inria.fr.
2. Since the AVI has dimension 2, it is also possible to find the solution by enumeration. A Matlab implementation of the solver by enumeration can found in [Huber, PhD thesis, 2015].
3. To sum up, the proposed controller is non-anticipative and the sliding variables $\widetilde{\Sigma}_{k+1} = (\widetilde{\sigma}_{k+1}, \widetilde{\dot{\sigma}}_{k+1})^T$ are always uniquely defined.

- Desired trajectory: $y_d = 40\sin(0.2\pi t) \quad (\text{in cm})$.
- Tracking error $e = y - y_d$.
- To measure the accuracy, we compute the average of the absolute value of the error over an interval of 60s. We call this quantity the *average tracking error* and we denote it by $\bar{e}$:

$$\bar{e} = \sum_{k=1}^{N} \frac{|e(t_k)|}{N} \quad \text{with} \quad t_N - t_1 = 60\text{s}.$$

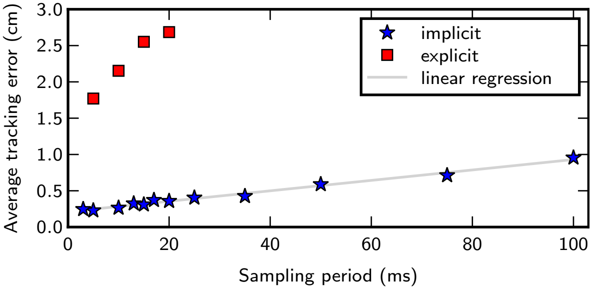


Figure: $\bar{e}$ w.r.t. h, implicit and explicit discretizations. Gain $G = 10^5$.

1. The explicit method yields unstable closed-loop for $h > 20$ ms.
2. Implicit method: error is $\mathcal{O}(h)$ while should be $\mathcal{O}(h^2)$: due to dirty differentiation for $\dot{y}$.

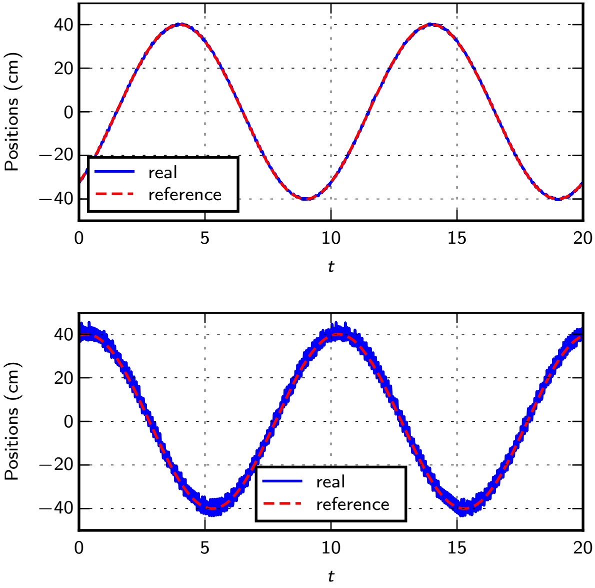
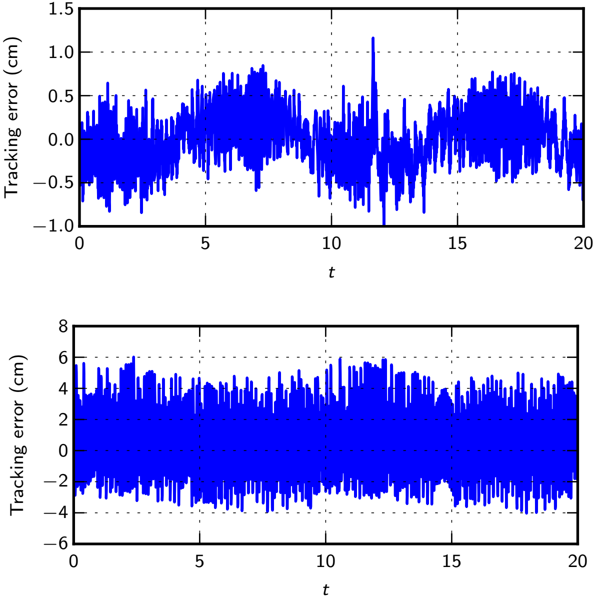


Figure: $h = 10\mathrm{ms}$, $G = 10^5$. Implicit (top), explicit (bottom).


Tracking error (cm)
1.5
1.0
0.5
0.0
−0.5
−1.0
0
5
10
15
20
t
Tracking error (cm)
8
6
4
2
0
−2
−4
−6
0
5
10
15
20
t

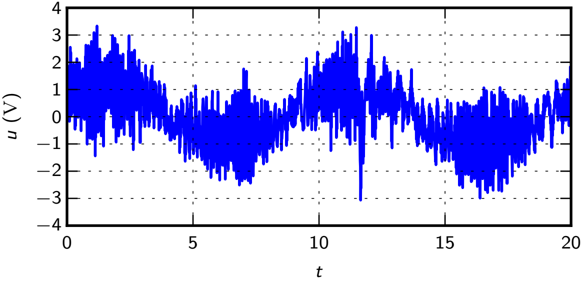


Figure: Control input u, $h = 10\mathrm{ms}$, $G = 10^5$, implicit discretization.

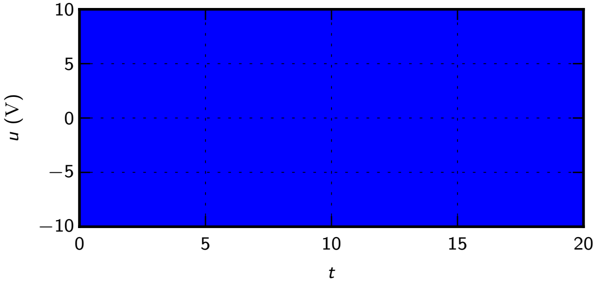


Figure: Control input u, $h = 10\mathrm{ms}$, $G = 10^5$, explicit discretization.

Augmenting the sampling time with the implicit method:

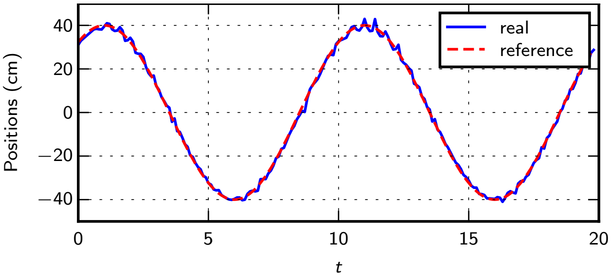


Figure: $h = 100\mathrm{ms}$ and $G = 10^5$.

The average tracking error is still better than with the explicit controller with $h = 10$ ms !

Insensitivity to the control gain G:

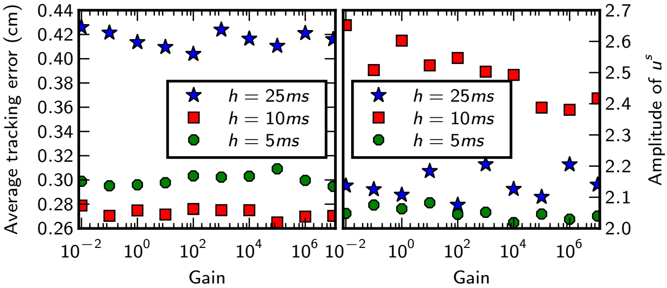


Figure: Evolution of the average tracking error and the control input amplitude when the gain G varies for 3 different sampling periods.

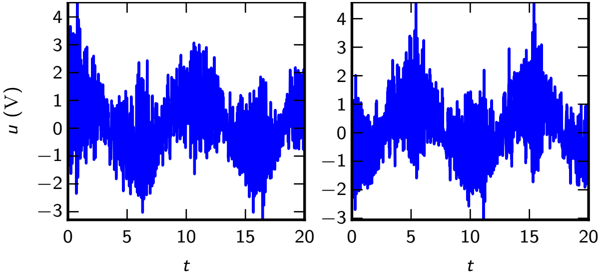


Figure: Control inputs for two values of the gain: 10^{-2} and 10^{7} and with a sampling period of $10\mathrm{ms}$.

$\rightsquigarrow$ *The tracking error and the input are insensitive to the control gain G.*

First order SMC vs. twisting:

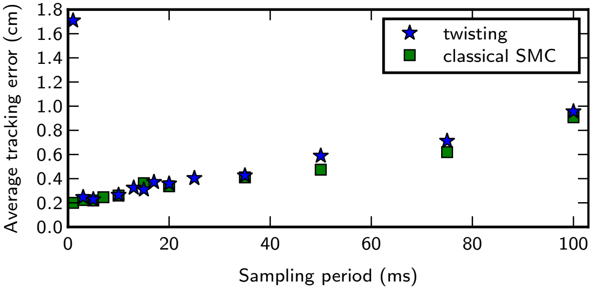


Figure: Comparison of the average tracking error with the implicit twisting and the implicit SMC, for $h \in [1, 100]\mathrm{ms}$.

$\rightsquigarrow$ *On this setup it seems that the first order SMC provides quite good performance which the twistng is not able to supersede.*

- The main features of the implicit controller are confirmed with the twisting algorithm:
  1. Allows for large sampling times without deteriorating the closed-loop behaviour.
  2. Controller and tracking error insensitive to control gain during the sliding phase.
  3. Chattering at both input and output is drastically reduced compared with the explicit method.
  4. The generalized equation to compute the controller, is a bit less easy than a projection but can still be solved with efficient methods and quickly.
- **But:** the selection of the parameters α (sliding variable) and of the two "dirty" differentiators, has to be done properly! See details in [Huber et al, CEP, 2016].

IMPLICIT DISCRETIZATION OF SIMPLE NONLINEAR SMC

We consider the scalar system which is globall fixed-time stable:

$$\boxed{\begin{aligned} &\dot{x}(t) = v(t) \\ &v(t) \in -x^3(t) - \mathrm{sgn}(x(t)) \\ &x(0) = x_0 \end{aligned}}$$

discretized as:

$$\boxed{\begin{aligned} &x_{k+1} = x_k - hx_{k+1}^3 + hu_{k+1} \\ &u_{k+1} \in -\mathrm{sgn}(x_{k+1}) \end{aligned}}$$

Explicit Euler controller fails to globally stabilize such a system and may yield instability [Efimov et al TACON 2017].

Two generalized equations with unknowns x_{k+1} and u_{k+1}, respectively:

$$\begin{cases} (1+hx_{k+1}^2)x_{k+1} - x_k + h\zeta_{k+1} = 0 \\ \zeta_{k+1} \in \mathrm{sgn}(x_{k+1}) \end{cases}$$

and

$$\begin{cases} \xi_{k+1} + h\xi_{k+1}^3 - x_k - hu_{k+1} = 0 \\ \xi_{k+1} \in -N_{[-1,1]}(u_{k+1}) \end{cases}$$

- Extends to the case with disturbance $\dot{x} = v(t) + d(t, x)$. The controller is then calculated from the GE:

$$\begin{cases} \tilde{x}_{k+1} = x_k + h u_{k+1} \\ u_{k+1} \in -\tilde{x}_{k+1}^3 - \mathrm{sgn}(\tilde{x}_{k+1}) \end{cases} \tag{35}$$

- All the above nice properties hold and the system is stabilized globally.
- Moreover the hyper exponential convergence rate is preserved after the discretization.

IMPLICIT DISCRETIZATION OF LTI SYSTEMS WITH PARAMETER UNCERTAINTY

The implicit method applies to systems:

$$\boxed{\begin{aligned}&\dot{x}(t) = (A + \Delta_A(t, x(t)))x(t) + B\big(u(t) + w(t, x(t))\big)\\ \\ &x(0) = x_0\end{aligned}}$$

where $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^m$, $w(t, x(t)) \in \mathbb{R}^m$ accounts for an external disturbance considered unknown but bounded in the L^∞ sense.

The addition of the term $\Delta_A(t, x)$ generates a nonlinear, time-varying, and state-dependent mismatched disturbance.

It is assumed that $\Delta_A(t, x)\Lambda\Delta_A^\top(t, x) < I_n$ for some known symmetric positive definite matrix $\Lambda \in \mathbb{R}^{n\times n}$.

- The set-valued control takes the form $-u^s \in K\sigma + \gamma(z)\mathbf{M}(\sigma)$ with $\mathbf{M}$ set-valued maximal monotone operator.
- **In continuous-time:** global asymptotic stability (and finite-time stability for the sliding-variable dynamics) is obtained with state-dependent gain $\gamma(z)$. With constant γ semi-global asymptotic stability is obtained.
- **In discrete-time:** we use constant γ and obtain semi-global practical stability, with convergence of the discrete-time solutions to a solution of the continuous-time differential inclusion.

BACK TO NUMERICAL ANALYSIS AND SIMULATION

The implicit Euler method also applies very well to circuits with ideal, nonsmooth, set-valued electronic components (diodes, transistors, etc).

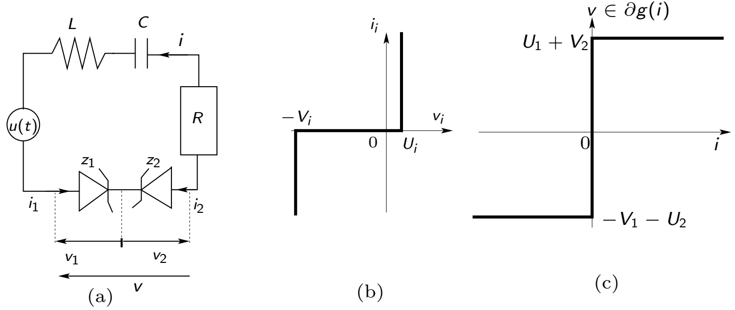


Figure: A circuit with two ideal Zener diodes in series.

The dynamics of such a circuit can be written as a Linear Complementarity System:

$$\begin{cases} \dot{x}(t) = Ax(t) + B\lambda(t) + Eu(t) \\ 0 \leq \lambda(t) \perp w(t) = Cx(t) + D\lambda(t) + Fu(t) + G \geq 0 \end{cases}$$

whose implicit Euler discretization reads:

$$\boxed{\begin{array}{l} x_{k+1} = x_k + hAx_{k+1} + hB\lambda_{k+1} + hEu_k \\ 0 \leq \lambda_{k+1} \perp w_{k+1} = Cx_{k+1} + D\lambda_{k+1} + Fu_k + G \geq 0 \end{array}}$$

This is a generalized equation (a Mixed LCP) which yields (provided that $I - hA$ is full rank) an LCP with unknown λ_{k+1} and LCP matrix $M_h \stackrel{\Delta}{=} D + hC(I - hA)^{-1}B$.

Indeed the complementarity conditions are rewritten as:

$$0 \leq \lambda_{k+1} \perp M_h \lambda_{k+1} + C(I - hA)^{-1}[x_k + hEu_k] + Fu_k + G \geq 0$$

From dissipativity properties the matrix M_h has good properties and this LCP can be solved at t_k.

A $\theta-$method can be used also.

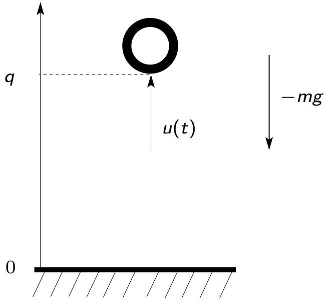
Remark: The mixed LCP can also be written equivalently as the inclusion:

$$\boxed{\begin{array}{l} x_{k+1} = x_k + hAx_{k+1} + hB\lambda_{k+1} + hEu_k \\ \\ \lambda_{k+1} \in -N_{\mathbb{R}^m_+}(w_{k+1}) \Longleftrightarrow w_{k+1} \in -N_{\mathbb{R}^m_+}(\lambda_{k+1}) \end{array}}$$

which is very close to some generalized equations we already met for the computation of set-valued controllers in SMC.

Another simple example: the bouncing ball with dynamics as:

$$\begin{cases} m\ddot{q}(t) = u(t) - mg + \lambda(t) \\ 0 \leq q(t) \perp \lambda(t) \geq 0 \\ \dot{q}(t^+) = -e_{\mathrm{n}}\dot{q}(t^-) \ \text{ if } \dot{q}(t^-) \leq 0 \text{ and } q(t) = 0. \end{cases}$$



We rewrite this dynamics as a second-order Moreau's sweeping process:

$$\boxed{m\,dv - (u(t) - mg)dt \in -N_{V(q)}(w(t))}$$

with

1. dv the differential measure of $v \overset{a.e.}{=} \dot{q}$,
2. $V(q)$ the tangent cone to $\{q \mid q \geq 0\}$,
3. $w(t) = \frac{v(t^+)+e_{\mathrm{n}}v(t^-)}{1+e_{\mathrm{n}}}$. When the velocity is continuous, $w(t) = v(t)$.
4. The set $N_{V(q)}(w)$ is the normal cone to the tangent cone, evaluated at w.

This is a measure differential inclusion.

The Moreau-Jean event-capturing time-stepping scheme is as follows:

$$\begin{cases} m(v_{k+1}-v_k)-h\,(u_k-mg)\in -N_{V(q_k)}(w_{k+1}) \\ q_{k+1}=q_k+hv_k \\ \\ \text{where}\ \ \lambda_{k+1}\in -N_{V(q_k)}(w_{k+1}),\ \ w_{k+1}=\frac{v_{k+1}+e_n v_k}{1+e_n} \end{cases}$$

which is a generalized equation for v_{k+1}.

We can rewrite it equivalently as:

$$\begin{cases} q_{k+1}=q_k+hv_k \\ \\ m(v_{k+1}-v_k)-h\,(u_k-mg)=0 & \text{if } q_k>0 \\ \\ m(v_{k+1}-v_k)-h\,(u_k-mg)=\lambda_{k+1} & \text{if } q_k\leq 0 \\ 0\leq \lambda_{k+1}\perp w_{k+1}=\frac{v_{k+1}+e_n v_k}{1+e_n}\geq 0 \end{cases} \tag{36}$$

We can further rewrite (36) as follows:

$$\begin{cases} v_{k+1} = v_k - \frac{h}{m}\left(u_k + mg\right) & \text{if } q_k > 0 \\ \\ m(v_{k+1} - v_k) - h\left(u_k + mg\right) = \lambda_{k+1} & \text{if } q_k \leq 0 \\ \underbrace{0 \leq \lambda_{k+1} \perp \frac{1}{m}\lambda_{k+1} + v_k - \frac{h}{m}\left(u_k + mg\right) + e_{\mathrm{n}} v_k \geq 0}_{\text{LCP with unknown } \lambda_{k+1}} & \end{cases}$$

An important point is that the above generalized equations (also called one-step non-smooth problem OSNSP) all take the generic form:

$$\boxed{0 \in f(x) + N_{\mathbf{C}}(x)}$$

with $\mathbf{C}$ convex, $f(\cdot)$ single-valued.

It can be formulated as quadratic programs, or complementarity problems, or variational inequalities, and be solved *efficiently* with suitable solvers.

- The implicit Euler or the Moreau-Jean algorithms allow to simulate complex nonsmooth mechanical systems (lots of DoF, lots of unilateral and bilateral contacts with set-valued friction)
- The implicit Euler implementation of SMC supersedes the usual explicit one in terms of chattering of both input and output (sliding variable).
- Implementations in SICONOS.
- Same ideas for higher order sweeping process $\approx$ switching DAEs with index r [Acary, Brogliato, Goeleven, Math. Prog. 2008].

SOME PERSPECTIVES

1. Developments of "easy-to-implement" pieces of code for GE solvers.
2. Twisting, super-twisting: preliminary results in [Acary et al TACON 2012]. To be completed.
3. Further comparisons between HOSM (discrete-time) and implicit first-order SMC (robustness, accuracy, chattering level of input and output). Preliminary results in [Koch et al 2016, Huber et al 2016], see also [Yan 2016].
4. Finite-time exact differentiators.

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